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COLLISION AND ESCAPE IN EINSTEIN'S PN FIELD

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SUMMARY: A two-body system is being considered in a PN formalism involving higher powers of variables and mixed terms. The McGehee-type coordinate transformations are used to blow up the collision singularity and to regularize the equations of motion. The collision manifold is obtained as a $2D$ torus embedded in the $4D$ phase space. A similar torus is obtained for the infinity manifold. The fictitious flows on these tori are described. They provide information about near-collisional orbits and about escape and near-escape orbits.

1. INTRODUCTION

There exists lots of quantitative studies about the two-body problem in the PN and even PPN approximations (see e.g. Brumberg 1972). As to qualitative investigations, there were performed studies on various models which describe the motion of a test particle subjected to the action of a PN field-generating source. Among such fields, we mention those of Manev-type (Maneff 1925; Diacu et al. 1995; Delgado et al. 1996) or Schwarzschild-type (Blaga and Mioc 1992; Moeckel 1992; Stoica and Mioc 1997).

The goal of this paper is to start an analysis that presents a twofold interest. On the one hand it offers to the mathematicians a physically more significant Hamiltonian for the study of the two-body problem. On the other hand, it provides to the physicists a very useful technique (familiar for the moment only to celestial mechanicians).

We shall consider the motion of a test particle in a spherical PN field with Einsteinian parametrization: $\alpha = 1,\ \beta = 1,\ \gamma = 1$ (see e.g. Soffel 1989). Choosing the units such that both the mass of the field-generating body and the Newtonian gravitational constant are equal to 1, and neglecting the terms of order c^{-4} and higher (c=speed of light), the Hamiltonian associated to the problem is:

$$H \equiv \frac{p_1^2+p_2^2}{2} - \frac{1}{\sqrt{q_1^2+q_2^2}} - \frac{\left(p_1^2+p_2^2\right)^2}{8c^2} - \frac{3\left(p_1^2+p_2^2\right)}{2c^2\sqrt{q_1^2+q_2^2}} + \frac{1}{2c^2\left(q_1^2+q_2^2\right)}, \tag{1}$$

where $\mathbf{q} = (q_1, q_2) \in \mathbf{R}^2 \setminus \{(0,0)\}$ is the position vector of the particle, whereas $\mathbf{p} = (p_1, p_2) \in \mathbf{R}^2$ is the conjugate momentum vector.

Observe that, due to the isotropy of space (e.g. Soffel 1989), the motion is confined to a plane. Without loss of generality, we shall consider that the

(q_1, q_2)-plane of the reference frame is the motion plane; this leads to the form (1) of the Hamiltonian.

In this first paper we shall focus on two limit situations of the motion: the collision and the escape to infinity.

The collision represents a singularity of the motion equations. Resorting to McGehee's (1974) technique, we blow up the singularity, and past instead of it, on the phase space, a manifold. The collision manifold proves to be a $2D$ torus embedded in the $4D$ space of the McGehee coordinates. Then we describe the flow on this manifold; the respective flow has no physical significance, but allows the understanding of the behaviour of near-collision orbits.

The study of near-infinity orbits starts from the motion equations in polar coordinates. By means of a Sundman-type rescaling of time, we obtain the infinity manifold, which also proves to be a $2D$ torus in the $4D$ space of the respective coordinates. The flow on this manifold is decribed, too. Physically, this flow is no more significant than the flow on the collision manifold, but it helps us to understand the asymptotic behavior of escape orbits, as well as the (escapeless) motion at very great distance from the source of the field.

To end, we emphasize the mathematical advantage of the study we start here. The Hamiltonian we deal with has a complex structure, involving both higher powers of the variables and mixed (coordinate-momentum) terms.

2. EQUATIONS OF MOTION AND FIRST INTEGRALS

The problem we tackle is described by the equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i},\ i = 1, 2, \tag{2}$$

with the Hamiltonian (1). The concrete form of these equations is

$$\dot{q}_i = p_i A,$$
$$\dot{p}_i = -\frac{q_i}{\left(q_1^2 + q_2^2\right)^{3/2}}\ B, \tag{3}$$

with $i = 1, 2$, and

$$A\left(\mathbf{q}, \mathbf{p}\right) = 1 - \frac{p_1^2 + p_2^2}{2c^2} - \frac{3}{c^2\sqrt{q_1^2 + q_2^2}},$$

$$B\left(\mathbf{q}, \mathbf{p}\right) = 1 + \frac{3\left(p_1^2 + p_2^2\right)}{2c^2} - \frac{1}{c^2\sqrt{q_1^2 + q_2^2}}. \tag{4}$$

Notice that the system is a conservative one, hence we may write the first integral of energy in the form

$$H = h, \tag{5}$$

where the constant of energy h represents the total mechanical energy of the system.

The angular momentum of the system is also conserved, providing another first integral:

$$q_1 p_2 - q_2 p_1 = K, \tag{6}$$

where K denotes the constant of angular momentum.

3. McGEHEE TRANSFORMATIONS

To tackle the collision, we leave the Hamiltonian character of the motion equations, and perform the McGehee-type transformations (McGehee 1974; see also Saari and Hulkower 1981). For the first step we pass to standard polar coordinates via the real analytic diffeomorphism

$$(q_1, q_2, p_1, p_2) \in \left(\mathbf{R}^4 \backslash \Delta\right) \longmapsto$$
$$(r, \theta, u, v) \in \left(\mathbf{R} \backslash \{0\}\right) \times \mathbf{S}^1 \times \mathbf{R}^2,$$

where $\Delta = \{(q_1, q_2, p_1, p_2) \mid q_1 = q_2 = 0\}$, while $\mathbf{S}^1$ is the segment $[0, 2\pi]$ with the end points pasted together. The explicit formulae are

$$q_1 = r\cos\theta,$$
$$q_2 = r\sin\theta,$$
$$p_1 = u\cos\theta - v\sin\theta, \tag{7}$$
$$p_2 = u\sin\theta + v\cos\theta.$$

The motion equations (3) acquire the form

$$\dot{r} = u\hat{A},$$
$$\dot{\theta} = \frac{v}{r}\hat{A},$$
$$\dot{u} = \frac{v^2}{r}\hat{A} - \frac{1}{r^2}\hat{B}, \tag{8}$$
$$\dot{v} = -\frac{uv}{r}\hat{A},$$

where (4) led to:

$$\hat{A}\left(r, u, v\right) = 1 - \frac{u^2 + v^2}{2c^2} - \frac{3}{c^2 r}, \tag{9}$$

$$\hat{B}\left(r, u, v\right) = 1 + \frac{3\left(u^2 + v^2\right)}{2c^2} - \frac{1}{c^2 r}.$$

The first integrals (5) and (6) become respectively:

$$\frac{u^2 + v^2}{2} - \frac{1}{r} - \frac{\left(u^2 + v^2\right)^2}{8c^2} - \frac{3\left(u^2 + v^2\right)}{2c^2 r} + \frac{1}{2c^2 r} = h, \tag{10}$$

$$rv = K. \tag{11}$$

Observe that both (8) and (10) keep the singularity, which now corresponds to $r = 0$.

For the second step, we use the real analytic diffeomorphism

$$(r, \theta, u, v) \in (\mathbf{R} \setminus \{0\}) \times \mathbf{S}^1 \times \mathbf{R}^2 \longmapsto (r, \theta, x, y) \in (\mathbf{R} \setminus \{0\}) \times \mathbf{S}^1 \times \mathbf{R}^2,$$

which scales down u and v through

$$u = \frac{x}{\sqrt{r}}, \quad v = \frac{y}{\sqrt{r}}. \tag{12}$$

Equations (8) transform into

$$\begin{aligned} \dot{r} &= r^{-1/2} x \tilde{A}, \\ \dot{\theta} &= r^{-3/2} y \tilde{A}, \\ \dot{x} &= r^{-3/2} \left[\left(\frac{x^2}{2} + y^2 \right) \right] \tilde{A} - \tilde{B}, \\ \dot{y} &= -\frac{1}{2} r^{-3/2} x y \tilde{A}, \end{aligned} \tag{13}$$

in which $\tilde{A}$, $\tilde{B}$ come from (9) as

$$\begin{aligned} \tilde{A}(r, x, y) &= 1 - \frac{x^2 + y^2 + 6}{2c^2 r}, \\ \tilde{B}(r, x, y) &= 1 + \frac{3\left(x^2 + y^2\right) - 2}{2c^2 r}. \end{aligned} \tag{14}$$

The first integrals (10) and (11) are now:

$$\frac{r\left(x^2 + y^2\right)}{2} - r - \frac{\left(x^2 + y^2\right)^2}{8c^2} - \frac{3\left(x^2 + y^2\right)}{2c^2} + \frac{1}{2c^2} = hr^2, \tag{15}$$

$$\sqrt{r} y = K. \tag{16}$$

The equations of motion are still singular. To remove this drawback, we have to rescale the time via the Sundman-type transformation:

$$dt = r^{5/2} d\tau \tag{17}$$

which bring the equations of motion (13) to the form:

$$\begin{aligned} r' &= r x A^*, \\ \theta' &= y A^*, \\ x' &= \left(\frac{x^2}{2} + y^2 \right) A^* - B^*, \\ y' &= -\frac{1}{2} x y A^*, \end{aligned} \tag{18}$$

where primes mark differentiation with respect to τ, and we kept by abuse the same notations for the new functions of τ. In formulae (18) the following notation was used:

$$\begin{aligned} A^*(r, x, y) &= r - \frac{x^2 + y^2 + 6}{2c^2}, \\ B^*(r, x, y) &= r + \frac{3\left(x^2 + y^2\right) - 2}{2c^2}. \end{aligned} \tag{19}$$

Also, we put the integral of energy (15) in the more convenable form

$$\frac{r\left(x^2 + y^2\right)}{2} - r - \frac{\left(x^2 + y^2 + 6\right)^2 - 40}{8c^2} = hr^2. \tag{20}$$

Observe now that the system (18) is well defined for $r = 0$, too. This means that the equations of motion were regularized with respect to the collision singularity. It is the same for the integral of energy.

Also notice that, due to the time rescaling (17), the (fictitious) time needed to reach the collision is infinite; in other words, the motion equations are now globally defined.

4. COLLISION MANIFOLD

McGehee's technique leads to regularized equations of motion; as a main result, it replaces the collision singularity with a manifold pasted on the phase space, as we shall see below. Although this manifold belongs now to the phase space, it is deprived of physical significance. However, due to the continuity of solutions with respect to initial data, the fictitious flow on the collision manifold provides valuable information about nearby orbits.

In our case the collision manifold is given by the set:

$$M_0 = \{(r, \theta, x, y) \mid r = 0 \text{ and (20) holds}\},$$

namely

$$M_0 = \left\{(r, \theta, x, y) \,| r = 0,\ \theta \in S^1;\ x^2 + y^2 = 2\left(\sqrt{10} - 3\right)\right\}. \tag{21}$$

The above set represents a $2D$ cylinder in the space of the coordinates (θ, x, y), or - since $\theta \in S^1$ - a $2D$ torus in the same space, both embedded in the full $4D$ space of the coordinates (r, θ, x, y).

It is easy to see that M_0 is invariant to the flow; indeed, by (18), $r' = 0$ for $r = 0$, therefore the full phase space extends smoothly to the boundary M_0.

Also notice that M_0 does not depend on the energy constant h. This means that every energy level shares this boundary.

To describe the flow on M_0, impose the conditions (21) to (19), which become $A^* = -\sqrt{10}/c^2$, $B^* = \left(3\sqrt{10} - 10\right)/c^2$. With these expressions equations (18) acquire the form

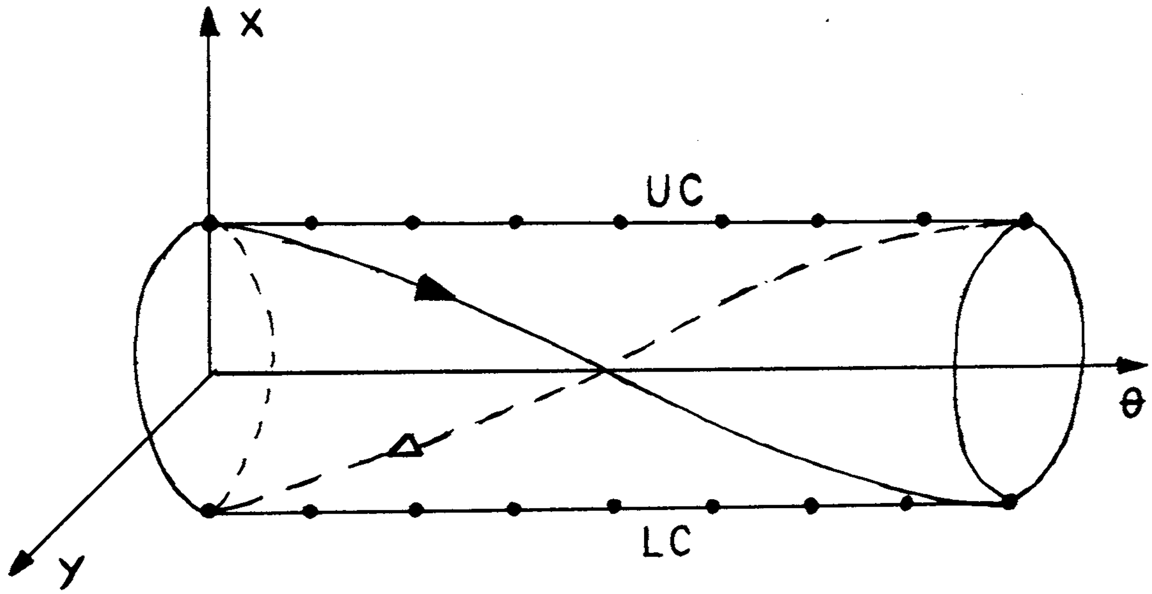

Fig. 1.

$$\theta' = -\left(\sqrt{10}/c^2\right) y,$$

$$x' = -\frac{1}{2}\left(\sqrt{10}/c^2\right) y^2, \tag{22}$$

$$y' = \frac{1}{2}\left(\sqrt{10}/c^2\right) xy.$$

The system (22) shows that there exist two circles entirely formed by degenerate equilibria on the M_0 torus: the upper circle UC defined by $\theta = \theta_0 \in S^1$, $x = -\sqrt{2\left(\sqrt{10}-3\right)}$, $y = 0$. Taking into account the second equation (22), one sees that all other orbits on M_0 are heteroclinic curves which start from UC and end in LC. As to their slope, putting $x = z\cos\varphi$, $y = z\sin\varphi$ (see for details Ballinger and Diacu 1993), we easily obtain $d\varphi/d\theta = -1/2$. Accordingly, the phase portrait on M_0 is plotted in Figure 1 (where the torus was represented as the initial cylinder).

Some remarks are to be made here. All orbits on M_0 tend asimptotically - in infinite (fictitious) time - to the stationary solutions LC. According to McGehee (1974), the flow on M_0 is gradient-like as regards the function $-x(\tau)$. Lastly, observe that the set of critical points $UC \cup LC$ is a compact set.

5. INFINITY MANIFOLD

In a certain sense, the opposite limit situation for the motion is the escape ($r \to \infty$). Nevertheless, there is an essential difference between this situation and the collision: the infinity is not a singularity for either motion equations or energy integral. In the sequel the infinity will also be transformed into a manifold, with the same goal as previously: the fictitious flow on it contributes to the understanding of the behaviour of bounded orbits which go very far from the field source.

There are two ways to reach this purpose; we shall follow the shorter one.

Denote $\rho = r^{-1}$, and rewrite equations (8):

$$\dot{\rho} = -\rho^2 u \hat{A}^*,$$

$$\dot{\theta} = \rho v \hat{A}^*,$$

$$\dot{u} = \rho v^2 \hat{A}^* - \rho^2 \hat{B}^*, \tag{23}$$

$$\dot{v} = -\rho u v \hat{A}^*$$

with, by (9),

$$\hat{A}^*(\rho, u, v) = 1 - \frac{u^2 + v^2 + 6\rho}{2c^2},$$

$$\hat{B}^*(\rho, u, v) = 1 + \frac{3\left(u^2+v^2\right) - 2\rho}{2c^2}. \tag{24}$$

The angular momentum integral (11) acquires the form

$$v = K\rho, \tag{25}$$

while the integral of energy (10) becomes

$$\frac{u^2+v^2}{2} - \rho - \frac{\left(u^2+v^2\right)^2}{8c^2} - \frac{3\left(u^2+v^2\right)}{2c^2}\rho + \frac{\rho}{2c^2} = h. \tag{26}$$

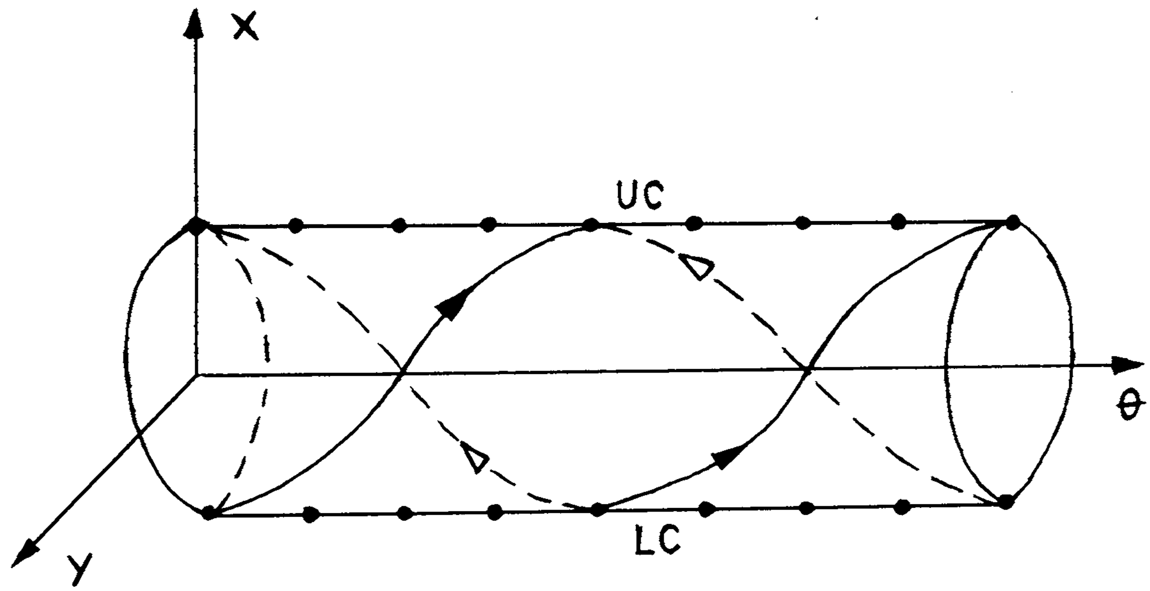

Fig. 2.

The next step is to rescale the time via

$$dt = \rho ds, \tag{27}$$

which transforms the motion equations (23) into

$$\begin{aligned} \frac{d\rho}{ds} &= -\rho u \hat{A}^*, \\ \frac{d\theta}{ds} &= v\hat{A}^*, \\ \frac{du}{ds} &= v^2\hat{A}^* - \rho\hat{B}^*, \\ \frac{dv}{ds} &= -uv\hat{A}^*. \end{aligned} \tag{28}$$

We define now the infinity manifold as

$$M_\infty = \{(\rho, \theta, u, v) \mid \rho = 0 \text{ and (26) holds}\},$$

that is,

$$\begin{aligned} M_\infty = \Big\{&(\rho, \theta, u, v) \mid \rho = 0;\ \ \theta \in S^1; \\ &u^2 + v^2 = 2c\left(c - \sqrt{c^2 - 2h}\right)\Big\}. \end{aligned} \tag{29}$$

This set is a $2D$ cylinder or a $2D$ torus (see Section 4) in the space of the coordinates (θ, u, v), both actually embedded in the $4D$ phase space (ρ, θ, u, v).

One observes that M_∞ is invariant to the flow (by (28), $d\rho/ds = 0$ when $\rho = 0$), hence the full phase space extends smoothly to the boundary M_∞.

With (29), the vector field (28) becomes

$$\begin{aligned} \frac{d\theta}{ds} &= v\hat{A}^*, \\ \frac{du}{ds} &= v^2\hat{A}^*, \\ \frac{dv}{ds} &= -uv\hat{A}^*, \end{aligned} \tag{30}$$

where, by (24) and (29), $\hat{A}^* = \sqrt{1 - 2h/c^2}$.

Examining (30), one sees that there are two circles of degenerate equilibria on the M_∞ torus: the upper circle UC defined by $\theta = \theta_0 \in S^1$, $u = \sqrt{2c\left(c - \sqrt{c^2 - 2h}\right)}$, $v = 0$, and the lower circle LC defined by $\theta = \theta_0 \in S^1$, $u = -\sqrt{2c\left(c - \sqrt{c^2 - 2h}\right)}$, $v = 0$. Putting, as in the previous section, $u = z\cos\varphi$, $v = z\sin\varphi$, one gets $d\varphi/d\theta = 1$. The phase portrait on M_∞ (taken as the initial cylinder) is plotted in Figure 2.

The same result could be obtained in a longer way (although more elegant). Starting from equations (18), with the prime integrals (16) and (20), we apply successively the transformations:

$$\rho = r^{-1}; \tag{31}$$

$$\xi = x\sqrt{\rho},\ \ \eta = y\sqrt{\rho}; \tag{32}$$

$$dw = -\rho^{-3/2} d\tau. \tag{33}$$

This leads to the same infinity manifold (given by the integral of energy) and to the same vector field on it. One observes easily that (ξ, η) are actually (u, v), and $dw = ds$.

To end, we notice by (25) that the infinity manifold can be reached only in the equilibria on UC. Also, the orbits which come from infinity can start only from LC.

6. CONCLUDING REMARKS

Some remarks can be made here:

(i) The energy level (given by h) is important for the infinity manifold, but does not affect the collision manifold. In other words, for every energy level there are orbits which start or end in collision. The situation is different as regards the infinity manifold. By (29), M_∞ is: a torus if $0 < h \leq c^2/2$; a circle if $h = 0$; the empty set if $h < 0$. This means that the orbits with negative energy levels are always bounded; the particle cannot escape.

(ii) It is necessary to emphasize the fact that each of the two manifolds carries its own (fictitious) time scale (differing from each other).

(iii) For $h > c^2/2$ the infinity manifold becomes meaningless. This fact confirms the physical background of the model within the framework of relativity.

(iv) The McGehee-type transformations proved to be a powerful tool in investigating the motion in such PN fields, even for these two limit situations only. The continuation of this research will prove again their efficiency.

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СУДАР И БЕКСТВО У АЈНШТАЈНОВОМ PN ПОЉУ

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Оригинални научни рад

Систем од два тела се разматра у оквиру PN формализма са вишим степенима промењљивих и мешовитим члановима. Координатне трансформације Мек Гехијевог типа се користе ради одстрањења сингуларности судара и ради регуларизације једначина кретања. Многострукост судара се добија као један $2D$ торус унутар фазног простора $4D$. Сличан торус се добија за многострукост бесконачности. Описују се фиктивни токови на овим торусима. Они обезбеђују информацију о скоро сударним путањама и о критичним (бекство), тј. скоро критичним путањама.