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UNPERTURBED TRAJECTORIES IN MANEFF'S GRAVITATIONAL FIELD ARE ELLIPSES IN VELOCITY PLANE

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SUMMARY: The unperturbed motion in Maneff's gravitational field (generated by a quasihomogeneous potential) is being considered. The trajectories (precessional conic sections) representing solutions of the Maneff two-body problem are found to appear just as (arcs of) conic sections in the velocity plane. The only case of astronomical interest, the (arcs of) ellipses one, is fully discussed and some remarks are made on qualitative behaviour of the motion.

1. INTRODUCTION

The general relativity theory answered many important questions in physics and astronomy; in particular it showed that the natural, unperturbed motion in the solar system is precessional (the trajectories are conic sections whose focal axes rotate in the plane of motion). Unfortunately, as regards the usefulness of such a tool for celestial mechanics, all attempts to formulate a meaningful relativistic n–body problem have failed to provide valuable results (see Diacu *et al.*, 1995). Thus, already in the twenties, there has appeared the necessity to create a gravitational model that could offer to astronomy the same answers as the relativity and equally respond to the theoretical needs of celestial mechanics.

Such a model, able to maintain the simplicity and the advantages of the Newtonian one, and also to provide the necessary corrections such that the orbits coming close to collisions match theory with observation, is hard to find. The post-Newtonian nonrelativistic gravitational laws generally failed, from an applicative astronomical standpoint, in explaining simultaneously certain questions as those concerning the secular motion of both perihelia of inner planets and Moon's perigee. Among such attempts there was however one exception: the law proposed by Maneff (1924, 1925, 1930 a, b) on the basis of physical principles, model which describes accurately both above issues, providing — at least at the solar system level — an equally good justification as the relativity (Hagihara, 1975).

Reconsidered very recently (Diacu, 1993, 1996; Diacu and Illner, 1993; Diacu *et al.*, 1995; Delgado *et al.*, 1995; Mioc and Stoica, 1995 a, b), Maneff's gravitational field appeared much less commonplace and much more unusual than at first sight, revealing new properties, very interesting and surprising at the same time. To give some supplementary examples in order to emphasize its importance, we mention that: as regards celestial mechanics, Maneff's case represents the only bifurcation of the flow among all quasihomogeneous potentials (Diacu, 1996), as well as the lowest order case for which the so-called ***black hole effect*** (nonrectilinear, but spiral collisions) occurs (Diacu *et al.*, 1995; Stoica and Stoica, 1995); as regards astrophysics, a Poisson-type equation corresponding to Maneff's potential would lead to new models of stellar interior and to new scenarios of stellar evolution (see Ureche, 1995); as regards theoretical physics, leaving aside the modeling of the Coulombian potential by a Maneff-type one (see Sommerfeld, 1951), an anisotropic Maneff model could contribute to a better understanding of the connections between classical and quantum mechanics (Diacu, 1993), and so forth. As one can see, this model offers a wide field of investigation covering various domains of research.

As proved by Krpić and Aničin (1993), the Keplerian trajectories of the Newtonian two-body problem are represented by circles (or arcs of circles) in the plane defined by the polar components of velocity. In this paper we treat the trajectories corresponding to the Maneff two-body problem in the same velocity plane. These ones prove to be (arcs of) conic sections (including hyperbolas and parabolas), but we dwell upon ellipses only, the sole case of astronomical interest.

2. EQUATIONS OF MOTION AND FIRST INTEGRALS

Consider the two-body problem in Maneff's field, generated by the potential (e.g. Diacu, 1993; Diacu *et al.*, 1995):

$$U = -\frac{Gm_1m_2}{r}\left[1+\frac{3G(m_1+m_2)}{2c^2r}\right] , \qquad (1)$$

where m_1, m_2 are the masses, r is the distance between them, G is the Newtonian gravitational constant, c is the speed of light.

The force field being central, the motion in the dynamic system (m_1, m_2) will be restricted to a fixed plane. Denoting $\mu = G(m_1+m_2)$ and using polar coordinates (r, u), it is easy to see that, with the potential (1), the relative motion of m_2, say, with respect to m_1, will be described by the equations (Mioc and Stoica, 1995 a, b):

$$\ddot{r} - r\dot{u}^2 + \frac{\mu}{r^2} + \frac{3(\mu/c)^2}{r^3} = 0 , \qquad (2)$$

$$r\ddot{u} + 2\dot{r}\dot{u} = 0 , \qquad (3)$$

with the following initial conditions:

$$(r,\ u,\ \dot{r},\ \dot{u})(t_0) = (r_0,\ u_0,\ V_0\cos\alpha,\ V_0\sin\alpha/r_0) , \qquad (4)$$

where V is the velocity and α is the angle between the initial radius vector and initial velocity.

Obviously, the angular momentum is conserved and (3) provides the first integral:

$$r^2\dot{u} = L , \qquad (5)$$

where $L = r_0V_0\sin\alpha$ is the constant angular momentum considered positive (direct motion). The first integral of energy can also be easily obtained by the usual technique:

$$V^2 = \dot{r}^2 + (r\dot{u})^2 = \frac{2\mu}{r} + \frac{3(\mu/c)^2}{r^2} + h , \qquad (6)$$

where $h = V_0^2 - 2\mu/r_0 - 3(\mu/c)^2/r_0^2$ is the constant of energy.

3. TRAJECTORIES IN VELOCITY PLANE

It is clear that the conservation of the angular momentum restricts the velocity space to a plane. So, let $V_r = \dot{r}$, $V_u = r\dot{u}$ be the polar components of the velocity. Putting V_u in (5), the latter reads

$$r = L/V_u . \qquad (7)$$

Substituting r given by (7) in (6), the latter becomes:

$$V_r^2 + \left[1 - \frac{3(\mu/c)^2}{L^2}\right]V_u^2 - \frac{2\mu}{L}V_u = h . \qquad (8)$$

Observe that we have assumed tacitly that $L \neq 0$. Indeed, for $L = 0$ (radial motion) we have $V_u = 0$ and the study of the trajectories in velocity plane becomes meaningless.

Equation (8) represents a family of conic sections in the (V_u, V_r) plane: hyperbolas, parabolas, or ellipses as $\left(L^2 - 3(\mu/c)^2\right)$ is negative, zero, or positive, respectively.

Let us make some remarks on the above cases from the astronomical point of view. Observe that the second term in (1) can be seen as a perturbing potential in a Newtonian two-body problem. As the "perturbing force" is central, recall that the focal parameter of the osculating conic sections remains constant all along the motion ($p = p_0$). Suppose that $L^2\ (= \mu p_0) \leq 3(\mu/c)^2$; this leads to $p_0 \leq 3\mu/c^2$, the latter quantity being of the order of Schwarzschild

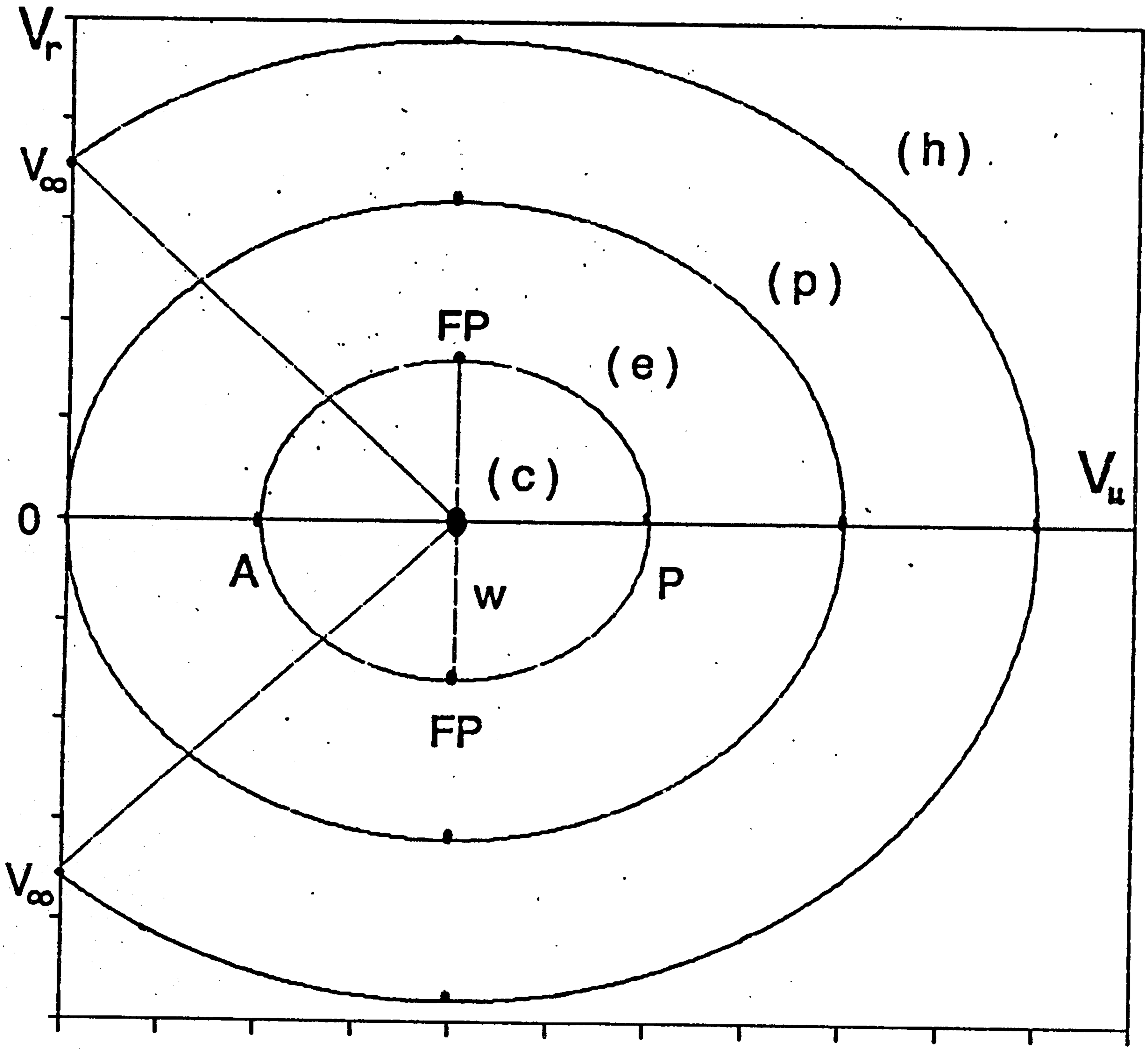

Fig. 1.

radius of the central body. It is obvious that this is not a realistic case among concrete astronomical situations, so we shall leave it aside and, consequently, we shall not study the hyperbolas and parabolas in the velocity plane.

For the family of ellipses represented by (8) for $L^2 > 3(\mu/c)^2$, let us denote

$$w = \frac{\mu L}{L^2 - 3(\mu/c)^2}, \quad a = w\sqrt{1 + \frac{hL}{\mu w}}, \quad b = \sqrt{h + \frac{\mu w}{L}}, \tag{9}$$

hence (8) acquires the form

$$\frac{(V_u - w)^2}{a^2} + \frac{V_r^2}{b^2} = 1 . \tag{10}$$

This is the equation of the generic ellipse of the family, whose center has the coordinates $(w, 0)$. The semimajor axis is a, the semiminor axis is b (both having the dimension of velocity), while the eccentricity is $e = \sqrt{1 - \mu/(Lw)} = \sqrt{3}\mu/(cL)$. Since in concrete astronomical situations $L^2 \gg 3(\mu/c)^2$, one sees easily that the ellipses (10) are very little eccentric. With L and h fixed by the initial conditions (4), w, a and b, given by (9), specify completely the ellipse.

Consider a point on the ellipse, and let the corresponding point on the director circle be defined by the polar angle ψ. The components of the velocity in the given point on the ellipse will be:

$$V_u = a\cos\psi + w\ , \quad V_r = b\sin\psi\ . \tag{11}$$

Replacing V_u given by (11) in (7), taking into account (9), and denoting $p^* = L/w$, $e^* = \sqrt{1+hL/(\mu w)} = a/w$, the expression (7) for the radius vector r becomes:

$$r = \frac{p^*}{1+e^*\cos\psi}\ , \tag{12}$$

which is the equation of a conic section of focal parameter (semilatus rectum) p^* , eccentricity e^* , and polar angle ψ. Indeed, as Diacu *et al.* (1995) showed, trajectories in the two-body problem for Maneff's field can be considered as precessional conic sections (namely conic sections whose focal axes rotate in the fixed plane of the motion); the apsidal motion is revealed by the fact that (cf. Diacu *et al.*, 1995):

$$\psi = \sqrt{\frac{\mu}{wL}}(u-u_0) + \arctan\frac{L\sqrt{wL}}{\sqrt{\mu}(L-r_0 w)\tan\alpha}\ . \tag{13}$$

It is easy to see that the angle ψ (which, by definition, may be considered as an equivalent of the eccentric anomaly for the elliptic "motion" in the (V_u, V_r) plane) is just the true anomaly for the motion on the precessional conic sections (at the initial instant $u = u_0$, $\alpha = \pi/2$, $\psi = 0$).

Let us now see how different types of precessional conic sections (12) are represented in the (V_u, V_r) plane (Figure 1).

If the trajectory is a circle (of radius p^*), we have $e^* = 0$, hence $h = -\mu w/L$ (the lowest possible value of the constant of energy) and (9) gives $a = b = 0$. The velocity diagram reduces to the point (c), for which $V = V_u = w$.

For precessional ellipses, $0 < e^* < 1$, hence $-\mu w/L < h < 0$ (the constant of energy is still negative, but is greater than that corresponding to the previous case), hence $a < w$. The velocity diagram is an ellipse (e) which does not touch the V_r–axis. At pericentre (P) the velocity is maximum $(V = V_u = w + a)$, and minimum $(V = V_u = w - a)$ at apocentre. For the points where the focal parameter touches the trajectory (FP), the radial velocity is maximum $(V_r = b)$, and we have $V_u = w$, $V = \sqrt{w^2+b^2}$.

For precessional parabolas, $e^* = 1$, $h = 0$, and (9) gives $a = w$; the corresponding ellipse (p) in the (V_u, V_r) plane is tangent to the V_r–axis. At pericentre $V = V_u = 2w$, while for FP the previous situation is recovered $(V_r = b,\ V_u = w)$. When $\psi \to \pi$, we have $V_u \to 0$, $V_r \to 0$.

Lastly, for precessional hyperbolas, $e^* > 1$, $h > 0$, and consequently, by (9), $a > w$. The corresponding velocity diagram (h) is an arc of ellipse restricted to the halfplane $V_u > 0$. In P we have $V = V_u = w + a$, in FP we have $V_r = b$, $V_u = w$. When $\psi \to \arccos(-w/a)$ we have $V_u \to 0$, $V_r \to (b/a)\sqrt{a^2-w^2}$.

Of course, all above characteristic values can also be expressed in terms of μ, c, L and h, or — more intuitively — as functions of μ, c and initial conditions.

Finally, let us make Maneff's potential tend to the Newtonian one, that is, let $c \to \infty$ in (1). In this limit, according to (9), $a = b = \sqrt{h+\mu^2/L^2}$, hence (10) is the equation of a circle of radius $\sqrt{h+\mu^2/L^2}$ and center $(w, 0)$ in the (V_u, V_r) plane. Leaving aside the differences of notation, we recover in this case the results obtained by Krpić and Aničin (1993).

4. CONCLUDING REMARKS

Reviewing the above results, one can formulate some conclusions concerning the qualitative behavior of the motion in Maneff's gravitational field.

In velocity plane all trajectories are (arcs of) conic sections. The families of hyperbolas and parabolas represent in terms of physical motion only collisional-type orbits (unbounded or bounded), but, as shown in Section 3, they do not reflect concrete astronomical situations. The ellipses in velocity plane correspond to stable motion for $h_c \le h < 0$ (circles for $h = h_c$, quasiperiodic or periodic orbits for $h_c < h < 0$), and to unstable motion of unbounded noncollisional-type for $h \ge 0$.

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НЕПЕРТУРБОВАНЕ ПУТАЊЕ У МАНЕФОВОМ ГРАВИТАЦИОНОМ ПОЉУ СУ ЕЛИПСЕ У РАВНИ БРЗИНА

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Разматрано је непертурбовано кретање у Манефовом гравитационом пољу (изведено од једног квазихомогеног потенцијала). Нађено је да се путање (прецесиони конусни пресеци), које представљају решење Манефовог проблема два тела, појављују само као лукови конусних пресека у равни брзина. Једини астрономски интересантан случај (елиптични лукови) потпуно је описан и дате су неке примедбе на квалитативан облик кретања.