Bull. Astron. Belgrade № 145 (1992), 25 – 31.

UDC 524.8-325
Original scientific paper

GEODESIC ORBITS IN ROSEN'S BIMETRIC GRAVITATION THEORY

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(Received: January 15, 1991)

SUMMARY: Some properties of geodesic lines in Rosen's bimetric gravitation theory (Rosen's spherically symmetric solution) are reconsidered. Results are compared with corrensponding properties of geodesic lines of Schwarzschild's solution in general relativity. Many similarities are identified. Equations of orbits are given, also, in their first order approximation. Approximative formulas for the shift of perihelia of planetary orbits and the deflection of light, in Rosen's theory, are determined.

1. INTRODUCTION

The subject of this paper is the motion of test particles in Rosen's spherically symmetric and static solution in his bimetric gravitation theory. That solution in the isotropic coordinates takes the form (Rosen, 1977):

$$ds^2 = -e^{-2M/r}\,dt^2 + e^{2M/r}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) \qquad (1)$$

In chapter 2 some analytical aspects of the equations of geodesic lines, as well as their geometrical shapes (obtained by numerical integration), were compared with corrensponding results of Schwarzschild's solution (in the isotropic coordinates too; see Brumberg, 1991) of general relativity. Chapter 3 is devoted to the case of the radial infall, while chapter 4 treats circular geodesic lines particulary. Finally, in chapter 5 equations of orbits are expanded in their first order approximation (weak field approximation). Approximative expressions for shift of the perihelia of planets and the deflection of light are determined.

2. QUALITATIVE ANALYSIS OF THE EQUATIONS OF GEODESIC LINES

In this chapter the equations of time-like and null geodesic lines are considered separately. Equations of time-like geodesics corrensponding to metrics (1) posses three first integrals:

a) integral of energy:

$$e^{-2M/r}\frac{dt}{ds} = \tilde{E}, \qquad (2)$$

b) integral of angular momentum:

$$r^2\frac{d\varphi}{ds}e^{2M/r} = \tilde{l}, \qquad (3)$$

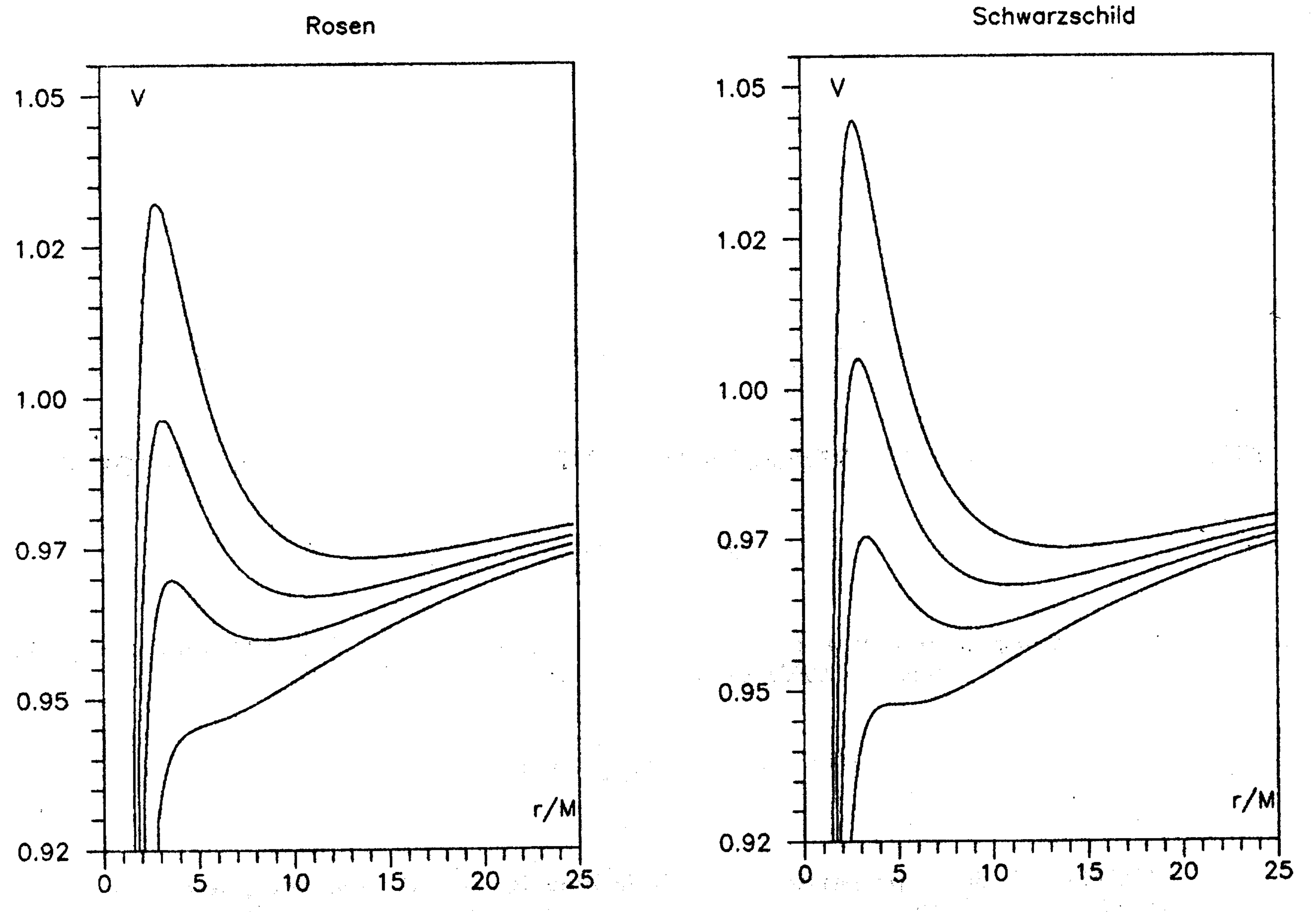

Fig. 1. Effective potentials in both theories

c) geodesic integral :

$$-e^{-2M/r}\left(\frac{dt}{ds}\right)^2 + e^{2M/r}\times$$

$$\left[\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\right] = -1, \tag{4}$$

where s denotes proper time (equations (2) through (4) are written in so called geometrical units $C = G = 1$). From first integrals one can see that the equations of motion on the hypersurface $\theta = \pi/2$ are:

$$\left(\frac{dr}{ds}\right)^2 = \tilde{E}^2 - e^{-2M/r}\left(1 + \frac{\tilde{l}^2}{r^2}e^{-2M/r}\right), \tag{5}$$

$$\frac{d\varphi}{ds} = \frac{\tilde{l}}{r^2}e^{-2M/r}, \tag{6}$$

$$\frac{dt}{ds} = \tilde{E}e^{2M/r}. \tag{7}$$

Effective potential is the function of the radial coordinate, which is used for qualitative analysis of the equations of type (5) (for Schwarzschild's solution of general relativity see Shapiro and Teukolsky, 1983.). Effective potential in this case is:

$$V_R(r) = e^{-M/r}\sqrt{1 + \frac{\tilde{l}^2}{r^2}e^{-2M/r}}. \tag{8}$$

The shapes of these functions (fig. 1), for different values of $\tilde{l}$, are very similar to the shapes of corrensponding curves in Schwarzschild's solution (in the isotropic coordinates). From that fact one can conclude that both types and shapes, of corrensponding geodesic lines are very similar in the two theories too.

On figures 2, 3 and 4 are presented examples of orbits: bound, unbound and orbits of capture (in two theories) respectively. Curves are obtained numerically by Hammings predictor-corrector method.

In case of null geodesic lines (orbits of light rays), instead of geodesic integral (4) one has:

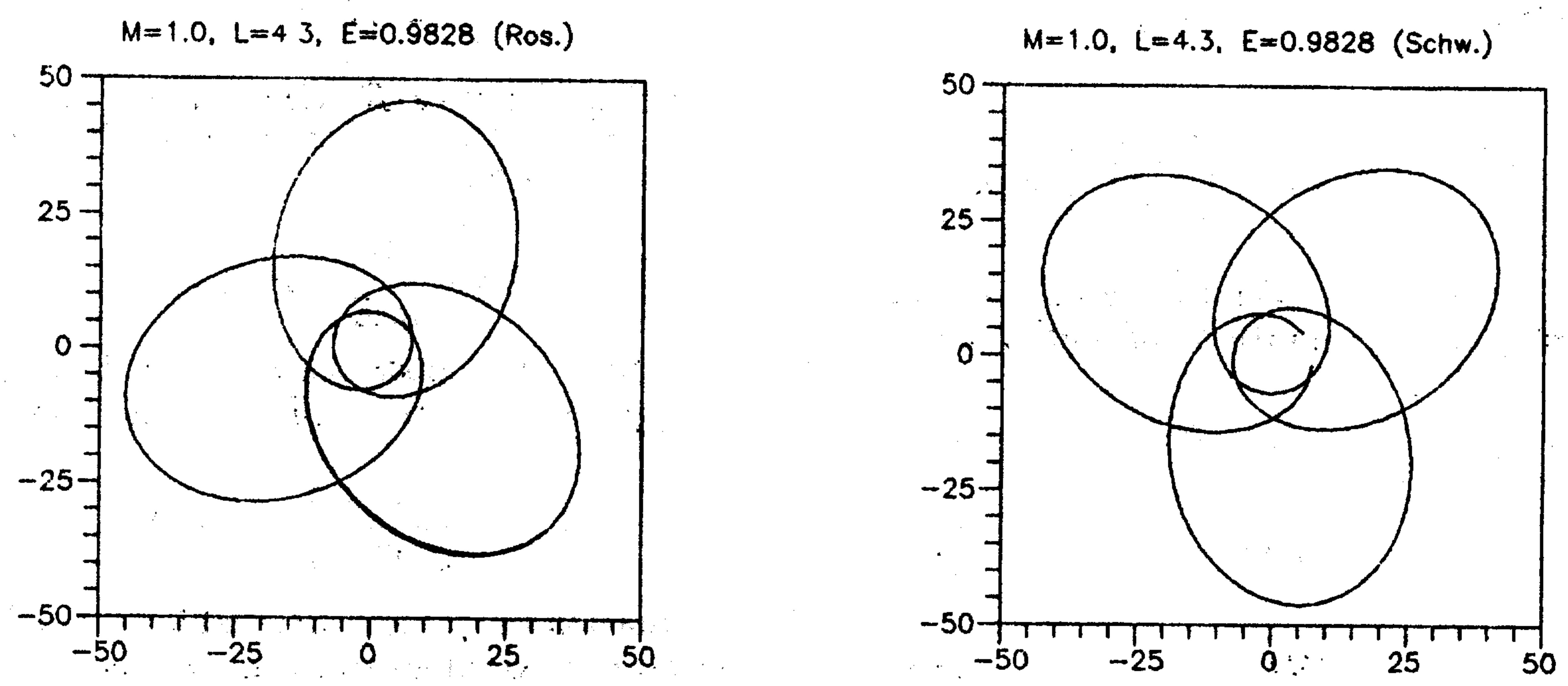

Fig. 2. **Example of bound orbits**

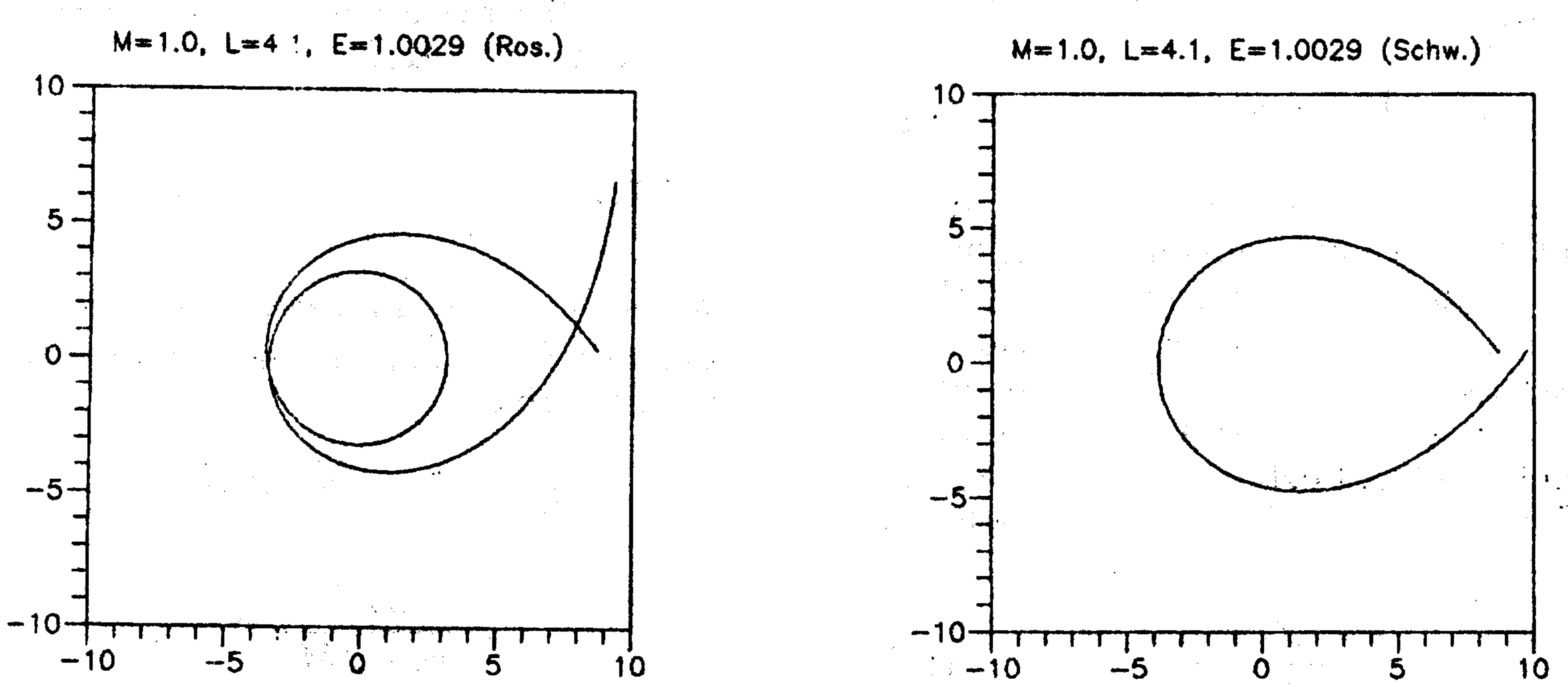

Fig. 3. **Example of unbound orbits**

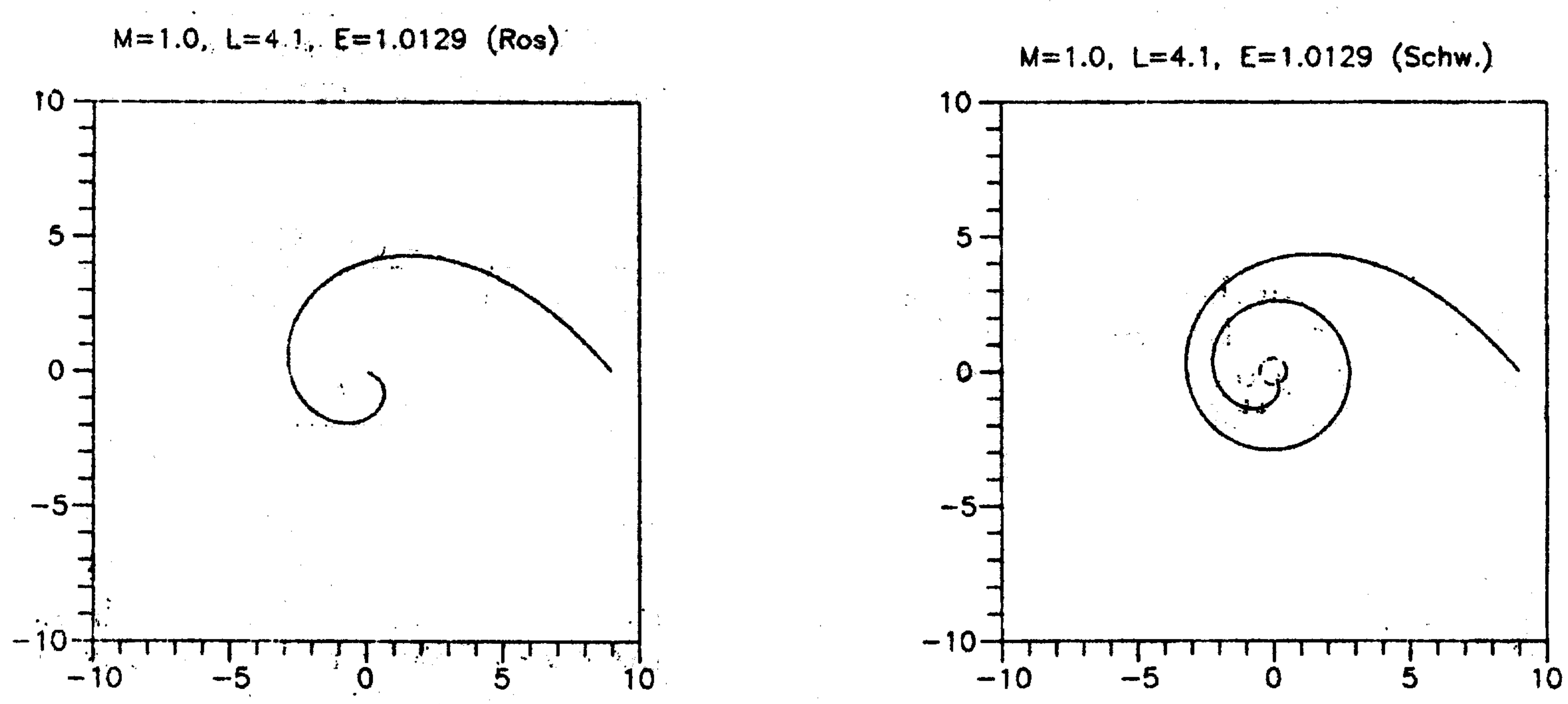

Fig. 4. **Example of orbit of capture**

$$-e^{-2M/r}\left(\frac{dt}{ds}\right)^2 + e^{2M/r}\times$$

$$\left[\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\right] = 0. \quad (9)$$

Integrals (2) and (3) take the same form in this case. Corrensponding equations of motion are:

$$\left(\frac{dr}{d\alpha}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2}e^{-4M/r}, \quad (10)$$

$$\frac{d\varphi}{d\alpha} = \frac{1}{r^2}e^{-2M/r}, \quad (11)$$

$$\frac{dt}{d\alpha} = \frac{1}{b}e^{2M/r}, \quad (12)$$

where $\alpha = \tilde{l}s$ and $b = \tilde{l}/\tilde{E}$. Effective potential in this case (equation (10)) is:

$$V = \frac{1}{r}e^{-2M/r}. \quad (13)$$

This function has only one extreme value (maximum) and therefore only unbound orbit or orbites of capture are possible (see Shapiro and Teukolsky, 1983). $V(r)$ has maximum at $r = M$ and that maximum takes value of $V_{max} = 1/eM$. That maximum evaluates, actually, the critical value (the value which divides unbound and orbits of capture) of the parmeter b. Effective potential in the Schwarzschild's solution (in the isotropic coordinates) is:

$$V = \frac{1}{r}\frac{1 - M/2r}{(1 + M/2r)^3}. \quad (14)$$

It has a critical value $b = 3\sqrt{3}M$ (as in standard coordinates) at the distance $r = M(1 + \sqrt{3}/2)$ (this radius corrensponds to circular orbit) from the source of the gravitational field.

3. RADIAL INFALL

Particulary interesting is case of the radial infall, because equations of motion are then much simpler. Test particle is falling directlly on the source of the gravitational field which means that angular momentum integral is zero $\tilde{l} = 0$. Those simplified equations, metioned above are:

$$\left(\frac{dr}{ds}\right)^2 = \tilde{E}^2 - e^{-2M/r}, \quad (15)$$

$$\frac{d\varphi}{ds} = 0, \quad (16)$$

$$\frac{dt}{ds} = \tilde{E}e^{2M/r}. \quad (17)$$

Critical distance R wich divides areas of the variable r where the motion is possible $r \leq R$ or impossible $r \geq R$ is:

$$R = -\frac{M}{\ln\tilde{E}} \quad (\tilde{E} < 1). \quad (18)$$

Appropriate critical distance in the Schwarzschild's solution (given in the isotropic coordinates) is:

$$R = -\frac{M}{2}\frac{\tilde{E}+1}{\tilde{E}-1} \quad (\tilde{E} < 1). \quad (19)$$

Equation (15) is not integrable by quadratures but the numerical integration shows (fig. 5) that for "coordinate observer" the particle is asymptotically falling on the source of gravity while in Schwarzschild's case for the same observer test particle never reaches distance $r = M/2$.

4. CIRCULAR ORBITS

Circular orbits occur in Rosen's solution (1) too. First integrals $\tilde{E}$ and $\tilde{l}$ are dependent upon the radius r_0 of circular orbit:

$$\tilde{E}^2 = \frac{r_0 - M}{r_0 - 2M}e^{-2M/r_0}, \quad (20)$$

$$\tilde{l}^2 = \frac{M{r_0}^2}{r_0 - 2M}e^{2M/r_0}, \quad (21)$$

respectively. According to formulas (20) and (21) the last possible circular orbit on the radius $r_0 = 2M$, corrensponds to the photon orbit ($\tilde{E} \to \infty$). In order to compare these results with the ones appearing in the Schwrzschild's solution one can derive appropriate expressions (in the isotropic coordinates) for that case:

$$\tilde{E}^2 = \frac{(2r_0 - M)^4}{(2r_0 + M)^2[M^2 - 8Mr_0 + 4{r_0}^2]}, \quad (22)$$

$$\tilde{l}^2 = \frac{M(M + 2r_0)^4}{4r_0[M^2 - 8Mr_0 + 4{r_0}^2]}. \quad (23)$$

Critical values of radius r_0 in this case are $r_{01} = M(1 - \sqrt{3}/2)$ and $r_{01} = M(1 + \sqrt{3}/2)$, but r_{01} is inside singularity ($r_{01} < M/2$), which means that the limiting case of the circular orbit corrensponds to r_{02}. Examples of circular orbits are presented on figure 6.

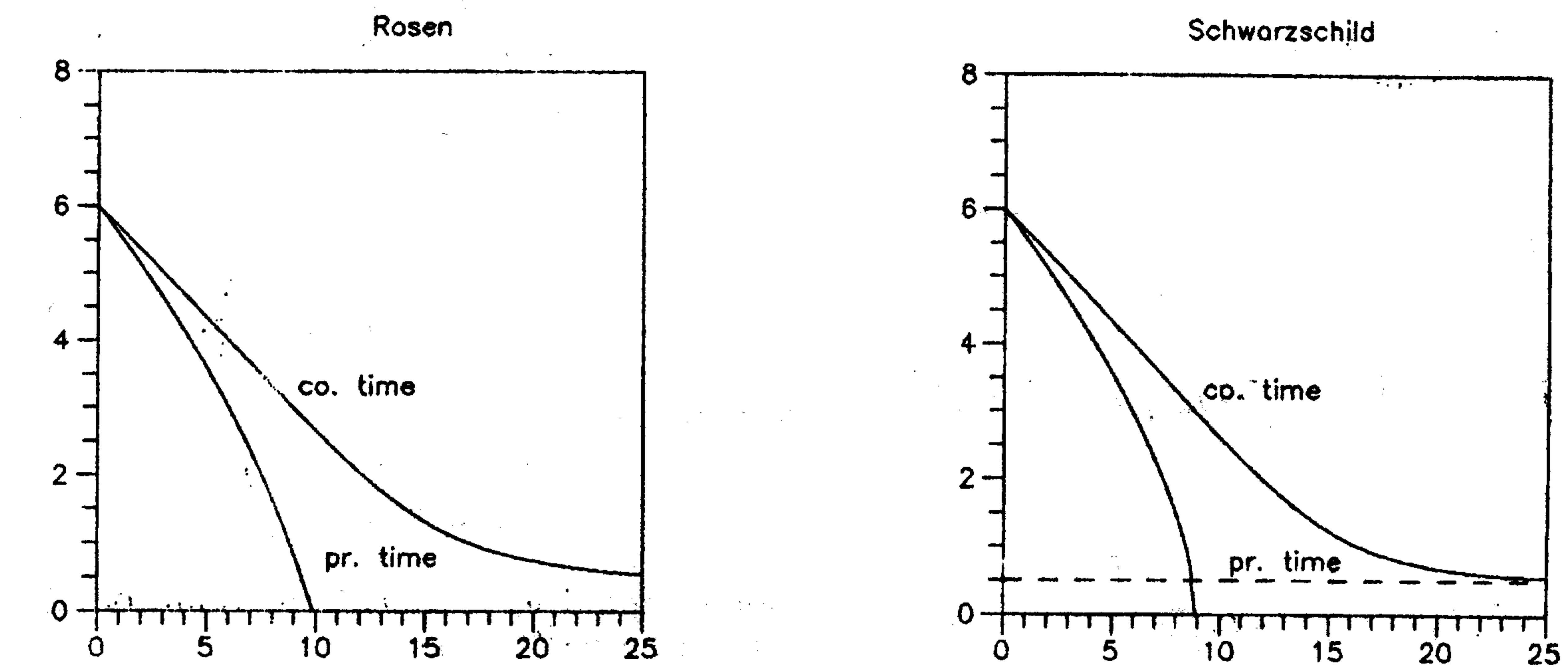

Fig. 5. Radial infall

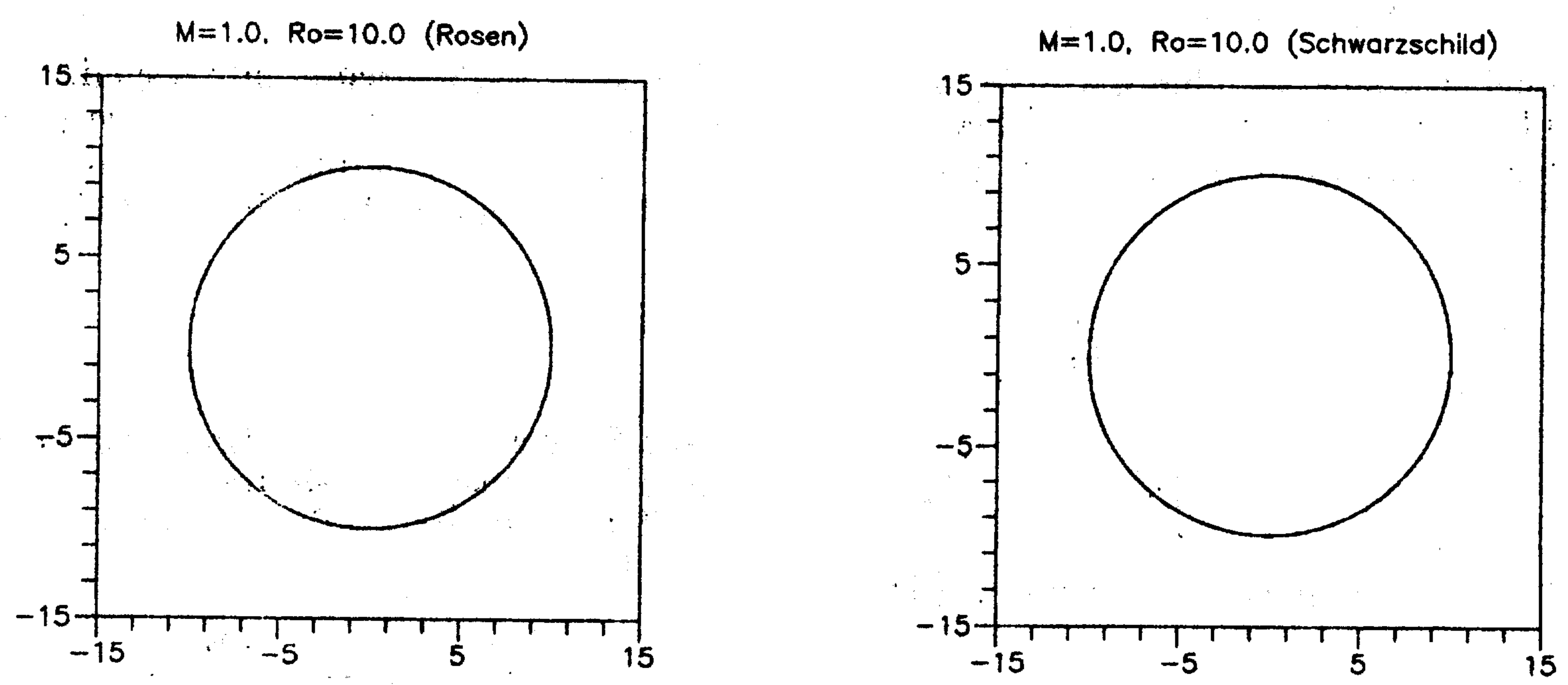

Fig. 6. Circular orbits

5. EQUATIONS OF ORBITS IN THE FIRST APPROXIMATION

From the analytical shape of the solution (1) Rosen (Rosen, 1977) himself, concluded that his spherically symmetric solution gives "the same agreement with present-day observations as the general relativity theory". Calculations done in this chapter show that he was right.

If one expands exponential functions in (1), and neglect quantities of the third order of M/r, the corrensponding approximative Lagrange function will be:

$$L_1 \equiv \left(1 - \frac{2M}{r} + \frac{2M^2}{r^2}\right)\left(\frac{dt}{ds}\right)^2 + \left(1 + \frac{2M}{r} + \frac{2M^2}{r^2}\right) \times \left[\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 + r^2 \sin^2\theta \left(\frac{d\varphi}{ds}\right)^2\right] = -1, \tag{24}$$

for time-like geodesic lines. Appropriate first integrals are:

a) equation (24),

b) integral of energy

$$\left(1 - \frac{2M}{r} + \frac{2M^2}{r^2}\right)\frac{dt}{ds} = \tilde{E}, \tag{25}$$

c) integral of angular momentum

$$r^2\left(1 + \frac{2M}{r} + \frac{2M^2}{r^2}\right)\frac{d\varphi}{ds} = \tilde{l}. \tag{26}$$

In that case equations of motion are:

$$\left(\frac{dr}{ds}\right)^2 =$$

$$\frac{\tilde{E}^2}{(1-2M/r+2M^2/r^2)(1+2M/r+2M^2/r^2)}-$$

$$\frac{1}{1+2M/r+2M^2/r^2}\left[1+\frac{1}{1+2M/r+2M^2/r^2}\frac{\tilde{l}^2}{r^2}\right], \qquad (27)$$

$$\frac{d\varphi}{ds} = \frac{1}{1+2M/r+2M^2/r^2}\frac{\tilde{l}}{r^2}, \qquad (28)$$

$$\frac{dt}{ds} = \frac{\tilde{E}}{1-2M/r+2M^2/r^2}. \qquad (28')$$

By introducing a new variable $u = 1/r$, and using φ instead of s as an independent variable, equation (26), if terms proportional to M^2u^2 are taken into account only, takes the form:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{\tilde{E}^2-1}{\tilde{l}^2} + \frac{2M}{\tilde{l}^2}\left(2\tilde{E}^2-1\right)u+$$

$$+\left[\frac{3M^2}{2\tilde{l}^2}\left(5\tilde{E}^2-1\right)-1\right]u^2. \qquad (29)$$

Furthermore, with the new parameters e and p (instead of $\tilde{E}$ and $\tilde{l}$)

$$\frac{e^2-1}{p^2} = \frac{\tilde{E}^2-1}{\tilde{l}^2}, \qquad (30)$$

$$\frac{1}{p} = \frac{M}{\tilde{l}^2} \qquad (31)$$

one can write (equation (29)):

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{e^2-1}{p^2} + \frac{2}{p}\left[1-\frac{2M}{p}(1-e^2)\right]u+$$

$$+\left\{\frac{2M}{p}\left[3-\frac{4M}{p}(1-e^2)\right]-1\right\}u^2. \qquad (31)$$

Parameters e and p are, actually, ecentricity and parameter of orbit, well known from classical celestial mechanics. Equation (31) can be simplified if one extracts the terms which are greater (some quantities were, already, neglected). For instance, for Earth, ratio M/p is about 10^{-8}, and therefore terms with $(M/p)^2$ can be neglected in this approximation. Simplified equation (31) takes the form:

$$\left(\frac{du}{d\varphi}\right)^2 + \left(1-\frac{6M}{p}\right)u^2$$

$$= \frac{e^2-1}{p^2} + \frac{2u}{p}, \qquad (32)$$

or

$$\frac{d^2u}{d\varphi^2} + \omega^2 u = \frac{1}{p}, \qquad (33)$$

where frequency ω is:

$$\omega \approx 1 - \frac{3M}{p} \qquad (34)$$

in an accepted approximation. Solution of equation (33) can be written as:

$$u = \frac{1+e\cos(\omega\varphi)}{p}. \qquad (35)$$

For $\omega = 1$, (35) represents a conical section, but for ω near to 1 pericentral distance (perihelion of planet) will slightly be moved after every passing through the pericenter. According to (34) that shift will be:

$$\delta\omega \approx \frac{6M\pi}{p},$$

like in the Schwarzschild's solution. Analogously to the case of time-like geodesic lines one can approximate null geodesic lines. The appropriate equation of orbit will be:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{1}{b^2} + \frac{4M}{b^2}u+$$

$$\left(\frac{8M^2}{b^2}-1\right)u^2, \qquad (36)$$

or after differentiation

$$\frac{d^2u}{d\varphi^2} + \omega^2 u = \frac{2M}{b^2}. \qquad (37)$$

In this case the frequency ω is:

$$\omega \approx 1 - \frac{4M^2}{b^2}. \qquad (38)$$

General solution of equation (37) could be written in the following form (Fock, 1956):

$$u = \frac{2M}{b^2} + \frac{1}{b}\cos\omega\varphi, \qquad (39)$$

where constants of integration are taken so that one has the minimal distance

$$r_{min} \approx b - 2M,$$

at $\varphi = 0$.

In Euclidian plane (r, φ) equation (39) determines a hiperbola (for ω one can take 1 in this approximation). Directions of the asymptotes of that hiperbola determine a small angle, which is actually light deflection. One can easily note that, in this case, that angle is:

$$\delta = \frac{2M}{b}, \tag{40}$$

as in Schwarzschild's solution.

Acknowledgments – I am indebted to Professor I. Lukačević for suggesting the subject of this study and for many fruitful discussions.

This work has been supported by the Republic Fund for science in Serbia through the project "Physic and Motions of Celestial Bodies".

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ГЕОДЕЗИЈСКЕ ОРБИТЕ У РОЗЕНОВОЈ БИМЕТРИЧКОЈ ТЕОРИЈИ ГРАВИТАЦИЈЕ

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УДК 524.8-325

Оргинални научни рад

Размотрене су неке особине геодезијских линија у Розеновој биметричкој теорији гравитације (Розеново сферно симетрично решење). Резултати су упоређени са одговарајућим особинама геодезијских линија у Шварцшилдовом решењу опште теорије релативности. Уочене су многе сличности. Једначине орбита су дате њиховом апроксимацијом првог реда. Изведени су приближни изрази за померање перихела планета и савијање светлости, у Розеновој теорији.