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ILLOGICALITIES IN ESTIMATING DISPERSION PARAMETERS BY MINQUE METHOD

K. Vračarić

Faculty of Civil Engineering University of Belgrade,
11000 Beograd, Bulevar revolucije 73, P.O.B. 895, Yugoslavia

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SUMMARY: Some illogicalities in theoretic foundation of MINQUE method used to estimate dispersion components are pointed out in the paper.

1. INTRODUCTION

Obtaining objective values of real quantities and also objective estimations of dispersion components is very important at elaborating astronomical measurements. MINQUE method is usually used to estimate dispersion components. Illogical foundation of this method will be pointed out in this work.

Theoretical consideration of MINQUE method for estimating dispersion parameters has been given in great number of magazines and in extensive professional literature. As a result of past studies of this method and its application, detailed theoretical derivation of formulas and theoretical proves are given.

I won't consider theoretical analysis and derivation of formulas, but critically review the final formulas and point out some illogicalities appearing in them.

When estimating dispersion components by the new method (Vračarić 1996), obtained values of tracks of covariance matrix are less than those obtained by MINQUE method. Hence, it can be concluded that the new method gives better results than MINQUE method.

2. REMARKS

In Karl-Rudolf Koch's book (1988.) in chapter 36^{th} conclusion can be found: "Thus, a *best quadratic unbiesed estimator* will be derived. According to (312.2) two conditions have to be fulfilled

$$1)\ \ E\left(\mathbf{h}^T\mathbf{D}\mathbf{h}\right) = \mathbf{p}^T\sigma \tag{1}$$

and 2) $\mathbf{V}(\mathbf{v}^T\mathbf{D}\ \mathbf{h})$ has to be minimum.
This is true, if and only if

$$\mathbf{DX} = 0 \tag{2}$$

$$V\left(\mathbf{h}^T\mathbf{D}\mathbf{h}\right) = 2\ tr\left(\mathbf{D}\ \mathbf{Q}\ \mathbf{D}\ \mathbf{Q}\right) \tag{3}$$

It is also proved that the unknown parameters are computed by formula

$$\mathbf{s} = \mathbf{S}^{-1}\mathbf{q} = \begin{bmatrix} S_{11} & S_{12} & ... & S_{1r} \\ S_{21} & S_{22} & ... & S_{2r} \\ ... & ... & ... & ... \\ S_{r1} & S_{r2} & ... & S_{rr} \end{bmatrix} \tag{4}$$

where:

$$S_{i,j} = tr(\mathbf{F}\ \mathbf{Q}_1\ \mathbf{F}\ \mathbf{Q}_j) \tag{5}$$

$$\mathbf{F} = \mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}(\mathbf{A}^T\mathbf{Q}^{-1}\mathbf{A})^{-1}\ \mathbf{A}^T\ \mathbf{Q}^{-1} \tag{6}$$

$$\mathbf{Q} = \mathbf{Q}_1 + \mathbf{Q}_2 + ... + \mathbf{Q}_r \tag{7}$$

$$q_i = tr(\mathbf{h}^T\ \mathbf{F}\ \mathbf{Q}_i\ \mathbf{F}\ \mathbf{h}) \tag{8}$$

$$\mathbf{R} = \mathbf{E} - \mathbf{A}(\mathbf{A}^T\ \mathbf{Q}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Q}^{-1} \tag{9}$$

$$\mathbf{F} = \mathbf{Q}^{-1}\ \mathbf{R} \tag{10}$$

Vector of the unknown $\mathbf{X}$ is not computed directly. Its value is present latently.

There holds:

$$\begin{aligned} &\mathbf{A}\ \mathbf{X} = \mathbf{M}(\mathbf{h}) \\ &\mathbf{X} = (\mathbf{A}^T\ \mathbf{A}^{-1}\ \mathbf{A})^{-1}\mathbf{A}^T\mathbf{Q}^{-1}\mathbf{h} \\ &\mathbf{V} = (\mathbf{E} - \mathbf{A}(\mathbf{A}^T\ \mathbf{Q}^{-1}\ \mathbf{A})^{-1}\mathbf{A}^T\mathbf{Q}^{-1})\mathbf{h} \\ &\mathbf{V} = \mathbf{R}\ \mathbf{h}\ , \qquad \mathbf{V}^T = \mathbf{h}^T\ \mathbf{R}^T \\ &\mathbf{F}\ \mathbf{h} = \mathbf{Q}^{-1}\ \mathbf{R}\ \mathbf{h} = \mathbf{Q}^{-1}\ \mathbf{V} \\ &\mathbf{h}^T\ \mathbf{F} = \mathbf{V}^T\ \mathbf{Q}^{-1} \end{aligned} \tag{11}$$

Matrixes $\mathbf{Q}$ and $\mathbf{F}$ are symmetrical.

It is clear that the value of the vector of the unknown $\mathbf{X}$ is present in the values or corrections $\mathbf{V}$. The values of corrections $\mathbf{V}$, unlike the independent measuring results, become mutually dependent through the process of leveling. Their dependence is expressed through the correlation matrix

$$\mathbf{Q}_v = \mathbf{R}^T\ \mathbf{Q}_h\ \mathbf{R} \tag{12}$$

But the dependence is not taken care of in further working process. Nor is taken care of the unequality between correlational matrixes of squares of corrections and linear values of correction. The affirmations formerly pointed out, will be proved in following derivation.

In order to shorten the derivation, the case of determing two dispersion components based on $\mathbf{n}$ measuring results will be considered, which will not lessen the generality of conclusions.

Covariance square matrixes will be denoted by $\mathbf{G}$ and $\mathbf{H}$. They can be fulfilled by all elements, but they are often diagonal. The case of diagonal square matrixes will be considered here. For schorter writing, elements on the main diagonal will be marked, unlike usually by G_{ii} and H_{ii}, by G_i and H_i

$$\mathbf{G} = \begin{bmatrix} G_1 & 0 & ... & 0 \\ 0 & G_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & G_n \end{bmatrix} ; \tag{13}$$

$$\mathbf{H} = \begin{bmatrix} H_1 & 0 & ... & 0 \\ 0 & H_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & H_n \end{bmatrix} .$$

The matric $\mathbf{F}$ is also symmetrical i.e. $\mathbf{F}_{ij} = \mathbf{F}_{ji}$.

$$\mathbf{F} = \begin{bmatrix} F_{11} & F_{12} & ... & F_{1n} \\ F_{21} & F_{22} & ... & F_{2n} \\ ... & ... & ... & ... \\ F_{n1} & F_{n2} & ... & F_{nn} \end{bmatrix} . \tag{14}$$

When the elements of the matrix $\mathbf{S}$ are written in developed form it is obtained

$$\begin{aligned} S_{11} &= tr(\mathbf{F}\ \mathbf{G}\ \mathbf{F}\ \mathbf{G}) = \\ &= F_{11}^2\ G_1^2 + F_{22}^2\ G_2^2 + ... + \\ &+ F_{nn}^2\ G_n^2 + 2\ F_{12}^2\ G_1\ G_2 + ... + \\ &+ 2\ F_{1n}^2\ G_1\ G_n + 2\ F_{23}^2\ G_2\ G_3 + \\ &+ ... + 2\ F_{2n}^2\ G_2\ G_n + ... \end{aligned} \tag{15}$$

$$\begin{aligned} S_{12} = S_{21} &= tr(\mathbf{F}\ \mathbf{G}\ \mathbf{F}\ \mathbf{H}) = \\ &= F_{11}^2 G_1 H_1 + F_{22}^2\ G_2\ H_2 + \\ &+ ... + F_{nn}^2\ G_n\ H_n + ... + \\ &+ F_{12}^2(G_1 H_2 + G_2 H_1) + \\ &+ F_{1n}^2(G_1 H_n + G_n H_1) + ... + \\ &+ F_{23}^2(G_2 H_3 + G_3 H_2) + ... + \\ &+ F_{2n}^2(G_2 H_n + G_n H_2) + ... \end{aligned} \tag{16}$$

$$\begin{aligned} S_{22} &= tr(\mathbf{F}\ \mathbf{H}\ \ \mathbf{F}\ \mathbf{H}) = \\ &= F_{11}^2 H_1^2 + F_{22} H_2^2 + ... + \\ &+ F_{nn}^2 H_n^2 + 2\ F_{12}^2 H_1 H_2 + ... + \\ &+ 2\ F_{1n}^2 H_1 H_n + 2\ F_{23}^2 H_2 H_3 + \\ &+ ... + 2\ F_{2n}^2 H_2 H_n + ... \end{aligned} \tag{17}$$

In order to write the elements of the matrix $\mathbf{q}$ in developed form it is necessary to point out the following requirements:

– The matrix $\mathbf{F}$ is symmetrical, given by Eq. (10) and by Eq. (11). Its transpose value is: $\mathbf{F} = \mathbf{F}^T = \mathbf{R}^T\mathbf{Q}^{-1}$;

– the matrix $\mathbf{Q}^{-1}$ is diagonal with elements

$$\mathbf{Q}^{-1} = \begin{bmatrix} p_1 & 0 & ... & 0 \\ 0 & p_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & p_n \end{bmatrix} . \qquad 18)$$

With the requirements just mentioned the elements of the matrix $\mathbf{q}$ are

$$\begin{aligned} q_1 &= tr(\mathbf{h}^T\mathbf{R}^T\mathbf{Q}^{-1}\mathbf{G}\ \mathbf{Q}^{-1}\ \mathbf{R}\ \mathbf{h}) \\ q_2 &= tr(\mathbf{h}^T\mathbf{R}^T\mathbf{Q}^{-1}\mathbf{H}\ \mathbf{Q}^{-1}\mathbf{R}\ \mathbf{h}) \end{aligned} \tag{19}$$

When (11) is substituted in (19), it is obtained

$$\begin{aligned} q_1 &= tr(\mathbf{V}^T\ \mathbf{Q}^{-1}\ \mathbf{G}\ \mathbf{Q}^{-1}\ \mathbf{V}) \\ q_2 &= tr(\mathbf{V}^T\ \mathbf{Q}^{-1}\ \mathbf{H}\ \mathbf{Q}^{-1}\ \mathbf{V}) \end{aligned} \tag{20}$$

Subsequent to the multiplications of matrixes indicated in Eq. (20) for elements of the matrix $\mathbf{q}$, finally is obtained

$$q_1 = p_1^2\ G_1\ V_1^2 + p_2\ G_2\ V_2^2 + ... + \\ + p_n^2\ G_n\ V_n^2 = [p^2\ G\ V^2] \qquad (21)$$

$$q_2 = p_1^2\ H_1\ V_1^2 + p_2\ H_2\ V_2^2 + ... + \\ + p_n^2\ H_n\ V_n^2 = [p^2\ H\ V^2]$$

In Gauss-Markoff model

$$\mathbf{A}\ \mathbf{x} = E\ (\ \mathbf{h}\ ) \qquad (22)$$

$$D\ (\ \mathbf{h}\ ) = \mathbf{Q} = \sigma_1^2\ \mathbf{T} + \sigma_2^2\ \mathbf{U} = \mathbf{G} + \mathbf{H}$$

matrixes **T**, **U**, **Q** and **D**(h) are diagonal and they can be written in non-matrix form

$$\begin{aligned} D(\ h_1\ ) &= \sigma_1^2\ T_1 + \sigma_2^2\ U_1 \\ D(\ h_2\ ) &= \sigma_1^2\ T_2 + \sigma_2^2\ U_2 \\ &\ldots\ldots\ldots\ldots \\ D(\ h_n\ ) &= \sigma_1^2\ T_n + \sigma_2^2\ U_n \end{aligned} \qquad (23)$$

The last expressions can now be written in a simplier matrix form

$$D\ (\ \mathbf{h}\ ) = \mathbf{B}\ \sigma = \begin{bmatrix} T_1 & U_1 \\ T_2 & U_2 \\ ... & ... \\ T_n & U_n \end{bmatrix} \cdot \begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \end{bmatrix} \qquad (24)$$

Dispersion $D(\mathbf{h}_i)$ in Eq. (23) is unknown. However, the values or corrections V_i are known, determined from by (first part of Gauss-Markoff model). They can be taken as an approximation in determing the estimation of dispersion. That's how we pass over to the equations of correction in the form

$$\begin{aligned} v_1 + V_1^2 &= \sigma_1^2 T_1 + \sigma_2^2 U_1 \\ v_2 + V_2^2 &= \sigma_1^2 T_2 + \sigma_2^2 U_2 \\ &\ldots\ldots\ldots\ldots \\ v_n + V_n^2 &= \sigma_1^2 T_n + \sigma_2^2 U_n \end{aligned} \qquad (25)$$

The procedure of determining correlation matrixes of square of correlation V^2 was shown in Vračarić (1996). When forming normal equations, from which the unknown estimations of dispersion components will be determined it is just, this correlation matrix that should be used, and not any other.

It can be noticed that coefficients of the matrix **S** given by Eqs. (15), (16) and (17) can be obtained as multiplication

$$\mathbf{S} = \begin{bmatrix} G_1 & G_2 & ... & G_n \\ H_1 & H_2 & ... & H_n \end{bmatrix} \cdot \qquad (26)$$

$$\cdot \begin{bmatrix} F_{11}^2 & F_{12}^2 & ... & F_{1n}^2 \\ F_{21}^2 & F_{22}^2 & ... & F_{2n}^2 \\ ... & ... & ... & ... \\ F_{n1}^2 & F_{n2}^2 & ... & F_{nn}^2 \end{bmatrix} \cdot \begin{bmatrix} G_1 & H_1 \\ G_2 & H_2 \\ ... & ... \\ G_n & H_n \end{bmatrix}$$

Elements of vector **q** can also be obtained as multiplication of matrixes

$$q = \begin{bmatrix} G_1 & G_2 & ... & G_n \\ H_1 & H_2 & ... & H_n \end{bmatrix} \cdot \qquad (27)$$

$$\cdot \begin{bmatrix} p_1^2 & 0 & ... & 0 \\ 0 & p_2^2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & 0 & p_n^2 \end{bmatrix} \cdot \begin{bmatrix} v_1^2 \\ v_2^2 \\ ... \\ v_n^2 \end{bmatrix}$$

3. RESULTS

When elements of the matrixes **S** and **q** are expanded and the essence of their meanings is appreheded (see Eq. (26) and (27) some illogicalties can be noticed. It is not clear how it can be theoretically proved that in forming matrix **S** one correlation matrix is used, and in forming the vector **q** an other correlation matrix, in both cases incorrect correlation matrixes.

It would be correct if, starting from equations of corrections (25), normal equations were formed in the form

$$\sigma = (\mathbf{B}^T\ \mathbf{Q}_{v^2}^{-1}\ \mathbf{B})^{-1} \mathbf{B}^T \mathbf{Q}_{v^2}^{-1}\ \mathbf{V}^2 \qquad (28)$$

where the correlation matrix of square of corrections is given in Vračarić (1996.)

$$\mathbf{Q}_{v_{ij}^2} = \mathbf{Q}_{v_{ij}}^2 \qquad (29)$$

and matrix $\mathbf{Q}_v$ shown by Eq. (12).

These are sufficient reasons why estimation of dispersion parameters by MINQUE method can not be accepted as theoretically correct.

The following conclusions can be made:

– values of dispersion components σ^2 will be obtained if equations of corrections (25) are used, in which the parameters σ^2, coefficients T_i and V_i, are unknown. In the following iterations new values of dispersion components tending to their final estimations, will be obtained

– if values of dispersion components, computed by formed iteration, are included for computing values of coefficients in equations of correction (25), some of the values of S^2 will appear as unknowns, and the equations of corrections will assume the form

$$\begin{aligned} v_1 &= s_1^2(\sigma_{1_s}^2 T_1) + s_2^2(\sigma_{2_o}^2 U_1) - V_1^2 \\ v_2 &= s_1^2(\sigma_{1_s}^2 T_2) + s_2^2(\sigma_{2_o}^2 U_2) - V_2^2 \\ &\ldots\ldots\ldots\ldots \\ v_n &= s_1^2(\sigma_{1_s}^2 T_n) + s_2^2(\sigma_{2_o}^2 U_n) - V_n^2 \end{aligned} \qquad (30)$$

When normal equations are formed from equations of correctrions (30) and the unknowns are determined from them, they will in iterations tend to values equal to one.

Mutual and alternate determination of unknown values by iterative process is performed because of correct determination of weights. By means of correct weights, values of the unknowns X are computed. Therefore the absurd is all the greater, to use incorrect values dispersion components for weights and correlation matrixes. Using incorrect weights does not change values of unknown drastically, but dispersion components, computed with wrong weights, will drastically differ from those computed with correct weights.

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НЕЛОГИЧНОСТИ КОД ОЦЕНЕ ПАРАМЕТАРА ДИСПЕРЗИЈЕ ПРИМЕНОМ MINQUE МЕТОДЕ

К. Врачарић

Институт за геодезију Грађевинског факултета, Булевар револуције 73, 11000 Београд, Југославија

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У раду се указује на неке нелогичности у теоријским поставкама MINQUE методе приликом њене примене за оцену компоненти дисперзије.