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GRAVITATIONAL RESONANCES

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SUMMARY: In this paper differential equation of relative rotation of a body in closed Keplerian orbit has been derived. Effects of the gravitational moment and of the orbit on this rotation are included by use of the shape factor and of the coupling function, respectively. Thereupon condition is proposed and resonance initial conditions are discussed. Finally, numerical simulation of Mercury's rotation is presented as an illustration of the obtained results.

1. INTRODUCTION

Orbital and rotational periods of certain solid heavenly bodies in closed Keplerian orbits are related as rational fractions. This phenomenon is termed resonance in celestial mechanics. Many authors have studied gravitational resonances in recent years (Garfinkel, 1982; Sergysels, 1989; Howland, 1989; de Moraes and da Silva, 1990; Beletskii, 1974; ...) Mathematically, the problem has been usually attacked by use of the perturbation technique. A different approach is used in this work. We hope that it provides a somewhat deeper insight into the problem.

The rotation of a body in the gravitational field is influenced by the gravitational moment (i.e., shape of the body) and by the orbital motion (shape of the orbit). Consequently, the shape factor is introduced and the function connecting the rotational and orbital motions was established first. Subsequently, the second order nonlinear differential equation of relative rotation of the body in gravitational field was derived.

Generally, if the shape factor is not zero, arbitrarily chosen initial conditions result in chaotic rotations of the body. However, it is possible to find some specific initial conditions generating nonchaotic rotational motions with periodic and coincident repetitions of disposition, angular velocity and angular acceleration. Such regular motions are termed resonance.

Knowing mass geometry of the body, one can plot dependence between the initial spin (initial coupling function value) producing chosen resonnance and eccentricity of the orbit. Maps containing sets of such resonance functions for two extreme values of the shape factor are given in this paper.

Numerical simulation of Mercury's rotation has been used to illusstrate the obtained results. An interesting effect has shown up. Due to strong nonlinearity of its coupling function, the great part of the perihelion half of the orbit Mercury passes with relative spin almost zero (resonance 1/1) and only because of its rotation on the aphelion half of the orbit, after two cycles, resonance 3/2 is attained.

2. SHAPE FACTOR, COUPLING FUNCTION AND DIFFERENTIAL EQUATION OF THE RELATIVE ROTATION

Consider a solid body moving in the closed orbit around, the dominant center of gravitation. Gravitational noise is excluded. The body revolves on the prIncipal axis (1) of its ellipsoid of inertia and this axis is at the right angle to the orbital plane. Any dimension d of the body is so small in comparison to the distance R from the center of mass to the center of gravitation that squares and higher powers of the fraction d/R may be neglected.

As is known, the orbital and rotational motions of a body in the gravitational field are coupled. However, while the influence of the orbit on rotation must be ignored, the influence of rotation on the orbit may be neglected if dimensions of the body are small in comparison with dimensions of the orbit and/or if the rotation is slow (Beletskii, 1974). In accordance with this assumption, it is taken in this work that the rotation of the body does not affect its orbit and consequently its orbital motion either.

The center of mass C was chosen to be at the origin of two moving systems of coordinates xCy and $\xi C\eta$ simultaneously. The former is related to the geometry of the orbit, Cx being oriented along R toward the center of gravitation. The orbital angle is ψ. The latter system of coordinates is related to the geometry of mass, $C\xi$ having direction of the principal axis (3) and $C\eta$ direction of the principal axis (2) of the ellipsoid of inertia. The position of the latter system of coordinates with respect to the former one is defined by the angle of relative rotation $\angle xC\xi = \varphi$ (Fig. 1).

If scales of length and time are taken to be a and $\sqrt{\frac{a^3}{GM}}$, where a is the semi-major axis of the (elliptical) orbit, G is the gravitational constant and M – mass of the center of gravitation, one gets Kepler's finite equation of the orbital motion

$$R = \frac{p}{1 + e\cos\psi} \tag{1}$$

and the second order differential equation governing the rotational motion of the body

$$\ddot{\psi} + \ddot{\varphi} = -\frac{s}{R^3}\sin 2\varphi\ , \tag{2}$$

in dimensionaless form.

In the equation (1) e represents the eccentricity and $p = 1 - e^2$ – the parameter of the ellipse. Every dot over the symbol on the left hand side of the equation (2) denotes one differentiation with respect to time. The expression on the right hand side of this equation is proportional to the gravitational moment and $s = \frac{I_2 - I_3}{I_1}$ is the shape factor (see Appendix). As seen, in the addopted model of motion, the shape factor may take values from 0 to 1.

Now we are going to transform the equation (2) by eliminating time from it and keeping the orbital angle as the only coordinate.

In the case of the central motion, the areal velocity is constant and related to the parameter of the orbit

$$\dot{\psi}R^2 = \sqrt{p}\ ,$$

so that, by use of the equation (1), the orbital angular velocity and acceleration may be represented in the following forms

$$\dot{\psi} = \frac{(1 + e\cos\psi)^2}{p^{3/2}} \tag{3}$$

and

$$\ddot{\psi} = -2e\sin\psi\frac{(1 + e\cos\psi)^3}{p^3}\ . \tag{4}$$

Using (3), the relative spin may be expressed as follows

$$\dot{\varphi} = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\dot{\psi} = \varphi'\dot{\psi}\ . \tag{5}$$

Equation (5) exposes the fact that the rotational and orbital motions are directly related at the angular velocity level. We shall call φ' the coupling function.

In the same way one can find that relative angular acceleration is equal

$$\ddot{\varphi} = \varphi''\dot{\psi}^2 + \varphi'\ddot{\psi}\ . \tag{6}$$

The introduction of relations (3), (4) and (6) into equation (2) leads to the differential equation of relative rotation

$$\varphi'' - 2e\frac{\sin\psi}{1 + e\cos\psi}(1 + \varphi') + s\frac{\sin 2\varphi}{1 + e\cos\psi} = 0\ , \tag{7}$$

which has to be accompanied by two initial conditions, of course

$$\varphi(0) = \varphi_0 \text{ and } \varphi'(0) = \omega_0\ . \tag{8}$$

We shall discuss these conditions in the next chapter.

From close examination of the equation (7) one can conclude that the second term in it represents the influence of the orbit and the third, the influence of the gravitational moment on rotation of the body.

The orbit always influences the rotation except in the case $e = 0$, that is, the case of circular central motion.

The gravitational moment always affects the rotation as well, excluding the case when the ellipsoid of inertia of the body is rotationally symmetric with respect to the figure axis (principal central axis (1)). In that case the shape factor is $s = 0$.

3. NECESSARY CONDITION AND RESONANCE INITIAL CONDITIONS

The orbital angular velocity (3) is maximal at the perihelion, $\psi_p = 2m\pi$, and minimal at the aphelion $\psi_a = (2m + 1)\pi$, (m=0,1,2,...). If the shape

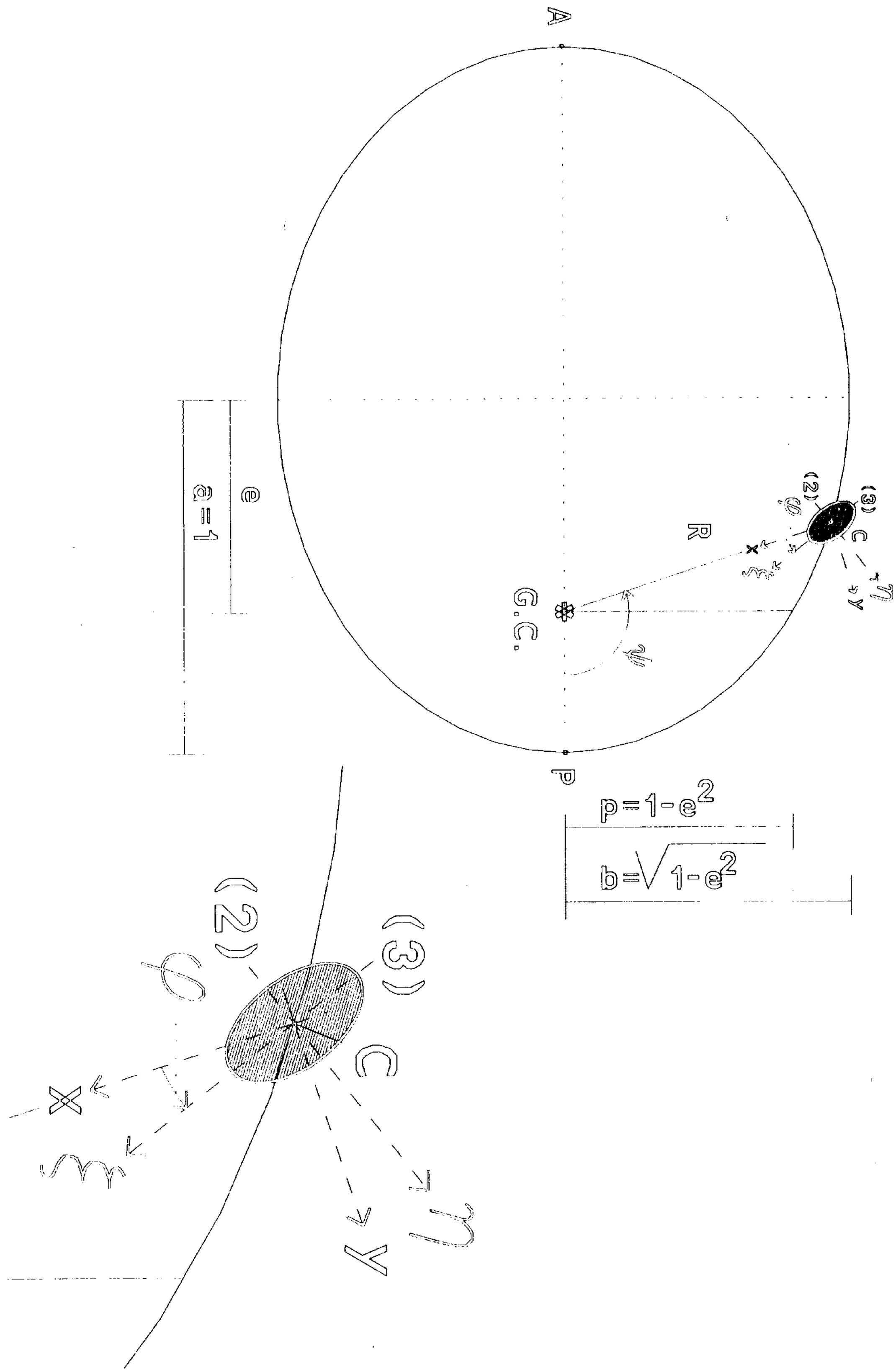

Fig. 1. *Disposition of two systems of coordinates.*

factor is $s = 0$, due to the coupling effect, the relative spin of the body is extreme at the same points (see eq. (7)). But it is also important to note that all rotational motions of such bodies are regular, that is, periodic.

On the other hand, when $s \neq 0$, arbitrarily chosen initial conditions produce generally chaotic rotations of the body. However, some specific initial conditions generate periodic and moreover, resonant rotational motions. If the shape factor of the body differs from zero, only resonant regular motions take place. All such rotations – there were no exceptions in our numerical analysis – were characterized by the extreme spin at the perihelion.

It seems that the following necessary condition may be established: regular rotational motion of the arbitrarily shaped body in closed Keplerian orbit is possible only if its relative spin has extremum at the perihelion. Consequently, relative angular acceleration $\ddot{\varphi}$ and likewise φ'' of the body must be zero at that point.

The analysis of the equation (7) shows that this condition may be fulfilled only if

$$\varphi(2m\pi) = \pm\frac{n\pi}{2} \quad (m, n = 0, 1, 2, ...)\ , \qquad (9)$$

where the minus sign stands for the case of the retrograde rotation.

If combined orbital-rotational motion of the body satisfies this relation, the rotation is regular, meaning periodic and simultaneous repetitions of disposition, spin and angular acceleration.

Since m and n are integers, it is worth noticing that relation (9) represents merely one formulation of resonance definition. Being a direct consequence of the introduced necessary condition, this result is the proof that it is correct.

Now the resonance ratio r may be calculated by dividing angles of absolute and orbital rotation after one orbital cycle

$$r = \frac{2\pi \pm \frac{n\pi}{2}}{2\pi} = \frac{4 \pm n}{4}\ , \quad (n = 0, 1, 2, ...) \qquad (10)$$

Evidently, depending on n, the resonance rotational motion may take place in one, two and four cycles.

Numerical simulations of rotational motions show that necessary condition produces extreme spin at the aphelion also, in all cases of one-cycle, as well as in certain cases of two-cycle resonances, but not in the other cases of two- cycle and in all cases of four-cycle resonances. Moreover, four-cycle resonances would never occur if the spin was extreme in aphelion (it seems that Earth's rotational and orbital motions are in four-cycle resonance).

Two possible explanations for such state of affairs are at our disposal.

The first one is that the influence of the orbit on the spin is lower in aphelion. Namely, part of the trajectory containing the aphelion the body passes with minimal and, depending on the eccentricity, almost constant orbital anglular velocity. Perhaps some kind of "uncoupling" takes place there. In the case of such stationary-like motion, the gravitational moment would have prevalent influence on the body rotation at the aphelion, producing dislocation of the extremum.

The second explanation is that four and two cycle resonances are merely transitory, less stable motions, on the way toward stable, one-cycle resonance motions of the body.

Let us consider now the initial values which produce resonant rotations of the body. The necessary condition (9) shows that, loosing nothing in generality of the solution, one resonance initial condition may be taken as homogenous, $\varphi_0 = 0$. Unfortunately, it is not so simple with the second initial condition, ω_0, which may be obtained only by trial and error method. For a given shape factor, the eccentricity of the orbit and a chosen resonance ratio, determination of the resonance initial spin (precisely, $\omega_0 = \varphi_0' = \frac{\dot{\varphi}_0}{\dot{\psi}_0}$ – resonance initial relative spin divided by initial orbital angular velocity) requires a series of numerical integrations of differential equation (7). It seems that convergence towards a satisfactory solution depends on stability of the resonance motion in consideration.

Fixing the shape factor s, one can find the relationship between the resonance initial spin ω_0 and the eccentricity of the orbit e, for any selected resonance ratio.

For the sake of systematization of the obtained results, two maps containing such functions are given in this work. The first one corresponds to the minimal shape factor $s = 0$ (Fig. 2) and the second to the maximal shape factor $s = 1$ (Fig. 3). For the practical use $0 < s << 1$ (see the following chapter), the case $s = 0$ is, in fact, trivial.

As can be seen, in the addopted space domain $\omega_0 \in (-3, 3)$, $e \in (0, 0.6)$, one cycle functions (heavy lines) taking resonance ratios from -9/1 to 19/1 appear.

As a rule, between two successive one-cycle resonance functions, at least one two-cycle and two four-cycle resonance functions are situated. Among these curves, -1/2, 1/2 and 3/2 two cycle functions (light lines) as well as -3/4 and -1/4 four cycle functions (dashed lines) were chosen to be presented in the maps.

The comparison of two maps shows that an increase of the shape factor produces displacements to the right and to the left of the functions with resonance ratios $r > 0$ and $r \leq 0$, respectively.

The increase of the shape factor mostly influences the form of the resonance functions corresponding to the small progressive spins. Thus, for instance, the single ideal resonance 1/1 curve in Fig. 2 is separated into two completely different curves in Fig. 3. Even more interesting is the bifurcation of the single resonance function 3/2 in the first map into three branches presented in the second map. Simmilar bifurcations should have been expected in the cases of 5/4 and 7/4 resonance functions.

In the beginning of the chapter it was mentioned that two arbitrarily chosen initial conditions generate chaotic rotational motions of the body with the shape factor $s \neq 0$. Nevertheless, we are aware

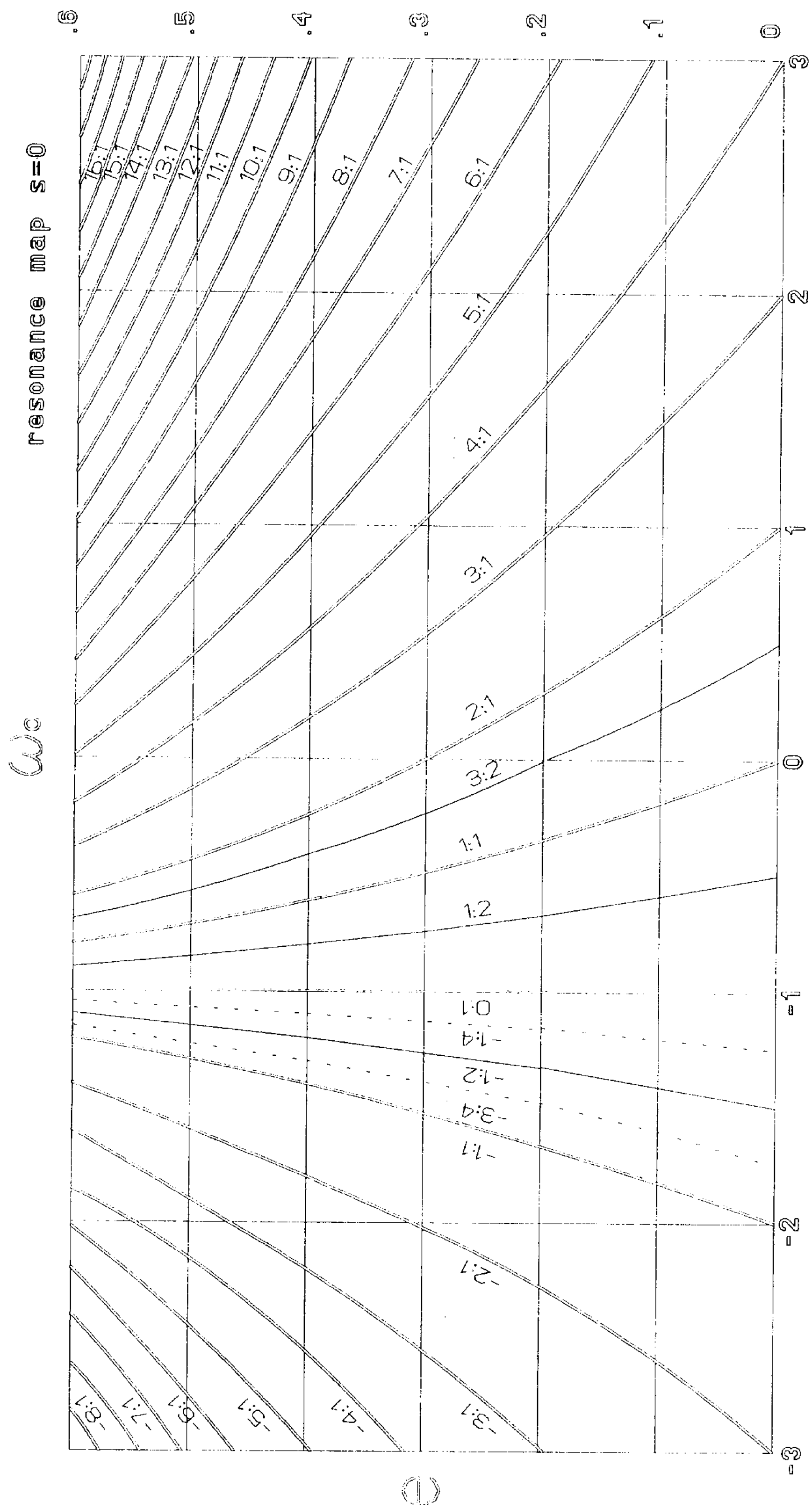

Fig. 2. *The map of the resonance initial conditions $\omega_0(e)$ for the shape factor $s = 0$.*

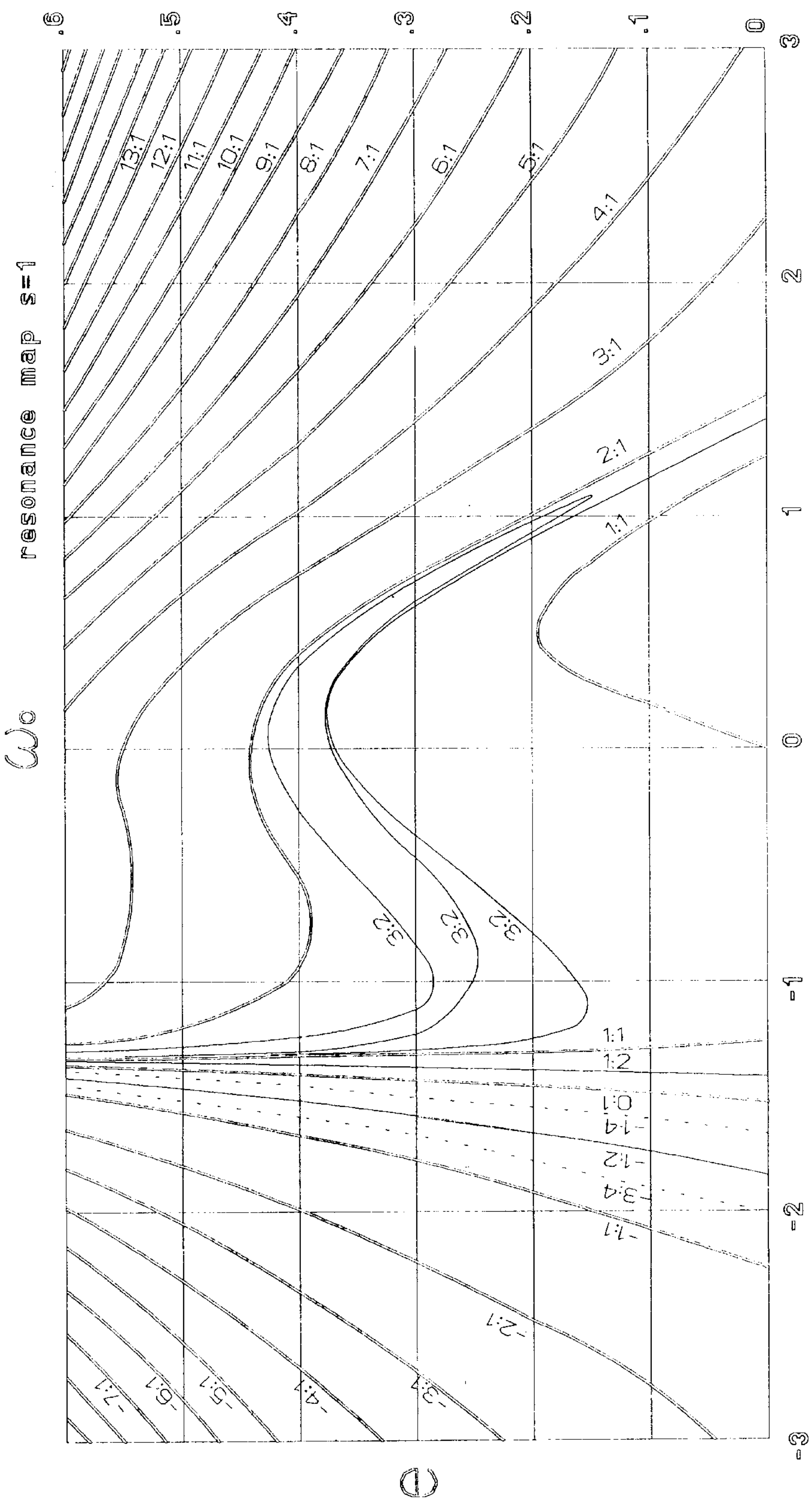

Fig. 3. *The map of the resonance initial conditions $\omega_0(e)$ for the shape factor $s = 1$.*

of the fact that rotational motions of the solid bodies in the closed orbits around centers of gravitation are mainly resonant. A natural question arises in what way chaotic rotational motion becomes converted into regular, resonant motion.

We propose the following explanation.

Given eccentricity of the orbit and the shape factor of the body, one can draw all curves $\varphi^r(\psi)$ representing resonant relative rotations, where r is the resonance ratio. On the same graph, the function $\varphi(\psi)$ representing chaotic rotation in consideration may be plotted as well. If that function, at any point located by the orbital angle ψ_0, touches (without intersecting) any one resonant function, the transition of the chaotic into corresponding resonant motion immediately takes place, that is, relative rotation of the body begins to follow the path of the touched function $\varphi^r(\psi)$, for $\psi \geq \psi_0$. In other words, values $\varphi_0 = \varphi(\psi_0)$ and $\omega_0 = \varphi'(\psi_))$ become initial conditions for the further resonant motion of the body. We can say that the touching point is the place where the body rotation is "locked" into the corresponding resonance.

Another question is whether one resonant motion may be converted into another or even into chaotic rotational motion. The answer is affirmative. If dissipative forces (as tidal forces are) act on the body, both rotational and orbital motions slow down and if these retardations are not commensurable with the resonance ratio, the change of the resonance rotation is inevitable. Whether this change leads into another resonance directly or through chaotic motion is yet to be seen. There is also possibility of collision of the body in consideration with some other body in the gravitational field. Depending on its magnitude, the impact may disturb resonant motion changing it into chaotic one.

4. ROTATION OF THE PLANET MERCURY

Numerical simulation of Mercury's rotation is given in this work as an example. The rotational axis of this planet is perpendicular to the orbital plane and that corresponds to the model addopted in this work. The eccentricity of the orbit of this heavenly body is 0.20563 and its resonance ratio is 3/2. It means that the planet makes three (absolute) rotations in the two orbital cycles. Radio tracking of the spacecraft Mariner 10 (1973) revealed the fact that the shape of this planet is close to the perfect sphere. Having no principal moments of inertia at disposal, we assumed in this paper that the shape factor is $s = 0.001$ (shape factor of Moon is $s = 0.084$, data taken from Beletskii (1974.)), noting that a ten times greater factor provides, approximately, identical results.

From the map in Fig. 2, for the corresponding eccentricity and resonance ratio, an approximate value ω_0 was taken first and then by trial end error method, more precise value $\omega_0 = -0.0347$ was determined. If the shape factor has been estimated exactly, one could say that Mercury has a small retrograde relative spin at perihelion. Anyhow, its perihelion spin is very close to zero. Relative rotation $\varphi(\psi)$ (heavy line) as well as the coupling function $\omega(\psi)$ (light line) of the planet through two orbital cycles are presented in Fig. 4.

The analysis of these graphs reveals an interesting fact: only 11.5% of the total relative rotation is accomplished along the perihelion half of the orbit and the rest, along the aphelion half. It means that the rotational motion along the perihelion half of trajectory is close to the ideal resonance 1/1. Positions of the planet in characteristic points of the orbit are schematically presented in Fig. 5.

If favourable observing conditions correspond to the perihelion or to the aphelion half of the orbit, such nonuniform rotational motion of the planet would inevitable lead to the wrong conclusion of its resonance ratio. That is, perhaps, the origin of the almost century long false belief (from Schiaparelli, 1882 to Pettengill and Buchanan, 1965) that Mercury is in the ideal 1/1 resonance.

5. CONCLUSION

The resonance between rotational and orbital motions of the solid body in closed Keplerian orbit were studied in this work.

The rotational motion is influenced by the shape of the body and by the shape of the orbit. Extreme relative spin in perihelion is the necessary condition for the regular rotational motion of the body. The only possible regular rotation of the arbitrarily shaped body in the gravitational field is resonant rotational motion. Resonances can take place in one, two and four cycles.

Numerical simulation of Mercury's resonant motion is given as an example. Relative rotation of this planet is quite nonuniform: only 11.5% of its total relative rotation is accomplished along the perihelion half of the orbit and the rest, along the aphelion half.

6. APENDIX: GRAVITATIONAL MOMENT

Keeping in mind all assumptions made at the start, we shall derive the expression for the gravitational moment with respect to the center of mass of the body. Positions of two moving systems of coordinates xCY and $\xi C\eta$ (Fig. 1) are related by the angle of relative rotation φ. According to Fig. 6, the intensity of the elementary gravitational force is, approximately

$$\mathrm{d}F = \frac{GM\mathrm{d}m}{(R-x)^2} \approx GM(\frac{1}{R^2} + \frac{2x}{R^3})\mathrm{d}m \ , \qquad (A1)$$

so that the gravitational moment with respect to the center of mass is

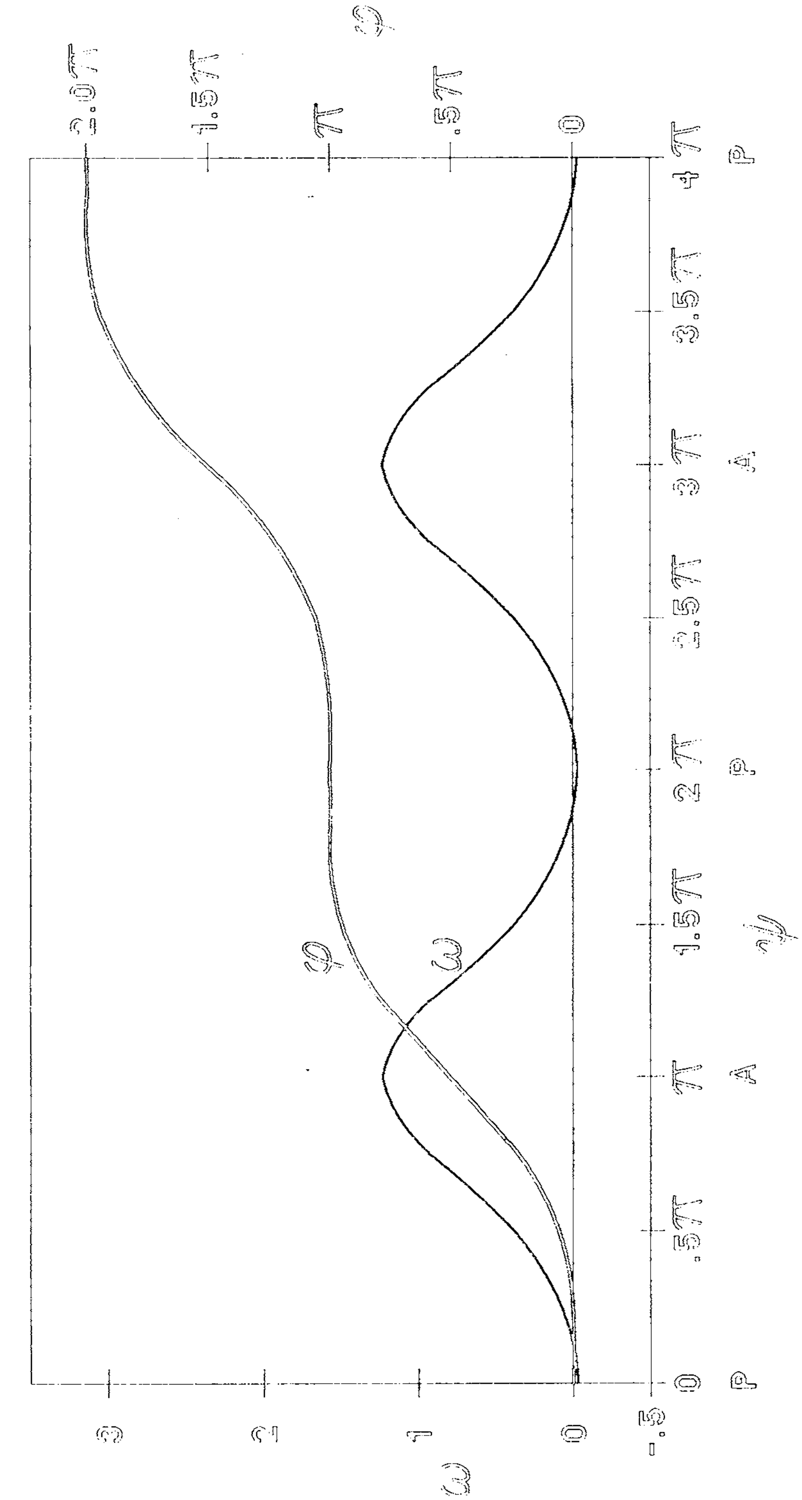

Fig. 4. *Relative rotation and coupling function of the planet Mercury.*

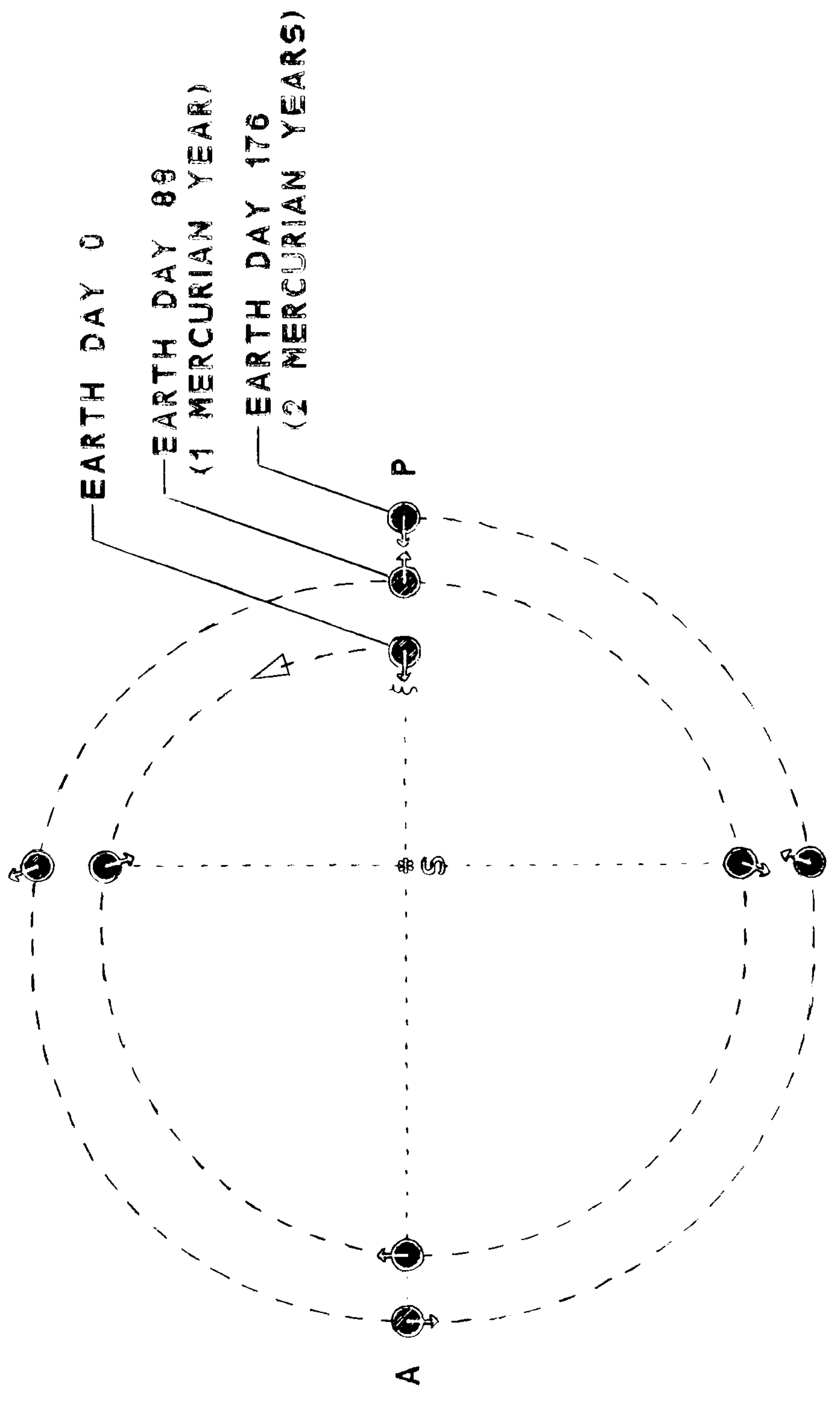

Fig. 5. *Positions of the planet at the characteristic points of the orbit.*

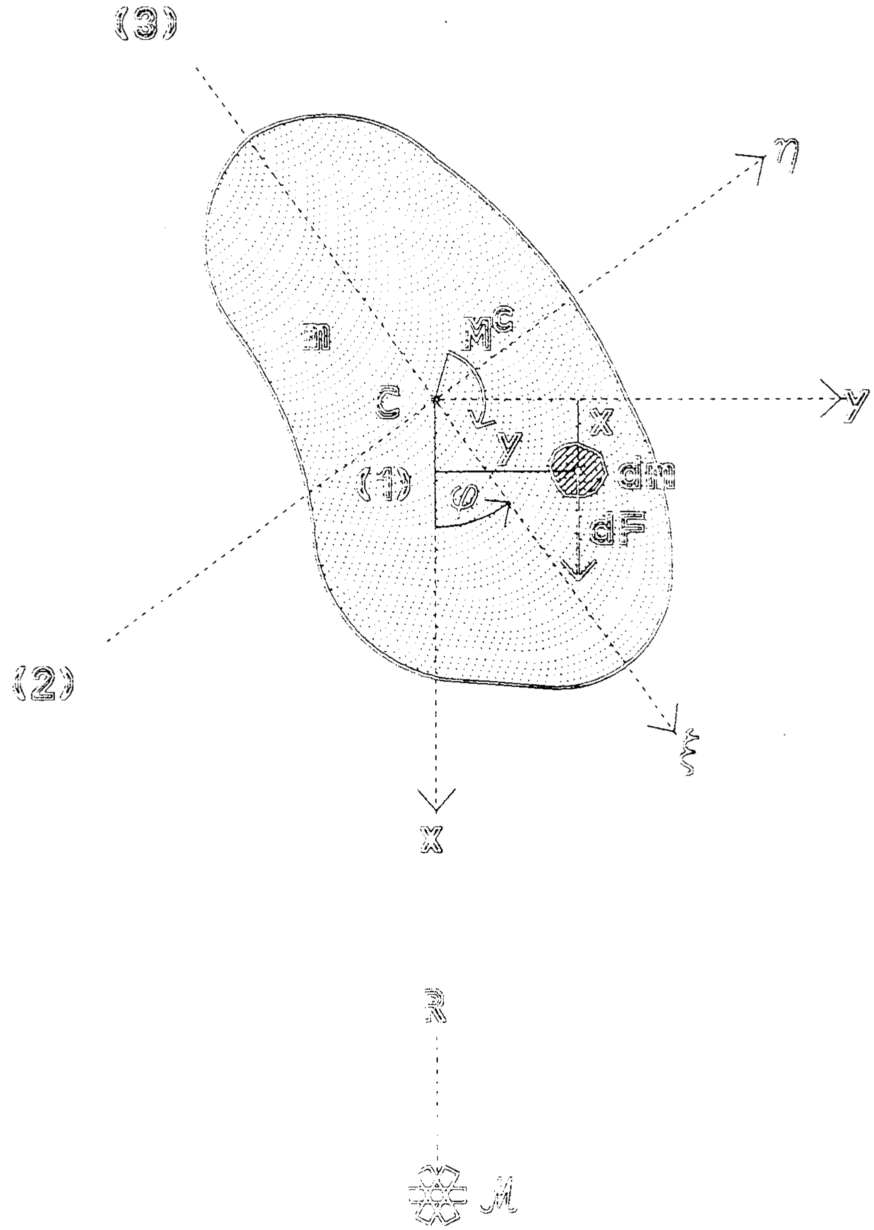

Fig. 6. *Gravitational moment with respect to the center of mass.*

$$M^c = -\int_m y \mathrm{d}F = -\frac{2GM}{R^3}\int_m xy\mathrm{d}m \; , \qquad (A2)$$

because $\int y\mathrm{d}m \equiv 0$.

By introducing coordinate transformation

$$x = \xi\cos\varphi - \eta\sin\varphi \; , \; y = \xi\sin\varphi + \eta\cos\varphi \qquad (A3)$$

one gets the final expression in the form

$$M^c = -\frac{GM}{R^3}\sin 2\varphi \int_m (\xi^2 - \eta^2)\mathrm{d}m = = -\frac{GM(I_2 - I_3)}{R^3}\sin 2\varphi \; . \qquad (A4)$$

In the dimensionaless form addopted in this work the gravitational moment is

$$M^c = -\frac{s}{R^3}\sin 2\varphi \qquad (A5)$$

In (A5) $s = \frac{I_2 - I_3}{I_1}$ is the shape factor and I_1, I_2 and I_3 are the principal central moments of inertia of the body ($I_1 \geq I_2 \geq I_3$).

The gravitational moment is a harmonic function of the angle of relative rotation and at the given point of the gravitational field, its amplitude depends on the shape of the body.

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У овом раду је изведена диференцијална једначина релативне ротације тела у затвореној Кеплеровој орбити. Утицаји гравитационог момента и орбита на ову ротацију су укључени помоћу фактора облика и функције везе, редом. Као илустрација добијених резултата дата је нумеричка симулација Меркурове ротације.