Bull. Astron. Belgrade № 156 (l997), 1 – 7

UDC 524.3–1/–16
Original scientific paper

THE EQUILIBRIUM OF POLYTROPES IN POLOIDAL MAGNETIC FIELDS

C. Blaga

Astronomical Institute of Romanian Academy
Astronomical Observatory Cluj
Cirećsilor Street 19, 3400 Cluj, Romania

(Received: May 7, 1997)

SUMMARY: In this article we study the equilibrium of polytropes in poloidal magnetic field. The magnetic field is the solution of a nonhomogeneous Euler equation for which we are able to give the general solution. It depends on the Lane–Emden function. We use the approximate analytic solution of the Lane–Emden equation proposed by Liu, to write it explicitly. For the polytropic indices $n = 1, 2$ the approximation is very good, the maximum error is $< 0.5\%$.

1. INTRODUCTION

The detection of the magnetic field at the stellar surface (see Babcock, 1958) raised the question of its role in the equilibrium of the star. Earlier, Chandrasekhar and Fermi (1953), using the virial theorem, found the maximum value for the magnetic field above which the equilibrium is lost. The measured values were below it, so the stars are in equilibrium. The problem of the magnetic field distribution in the star gives rise to a complicated mathematical problem. An analytical solution of the problem could be found only if we would make some supplementary hypotheses. We shall presume, based on the photometric observations (Borra and Landstreet, 1980), that at the stellar surface the magnetic field can be approximated by a dipole with the center in the center of the star and so the field has an axial symmetry. We suppose that the field is weak. We also consider that the star is a polytrope with a given polytropic index n. In this case to obtain the distribution of the magnetic field in the star we have to solve a singular Sturm-Liouville problem (see Roxburgh, 1966). In the particular case of the poloidal magnetic field (see the decomposition proposed by Lüst and Schlüter, 1954), the problem can be easily solved, because it yields a nonhomogeneous Euler equation. Its general solution will depend on the Lane–Emden function.

The equilibrium of a polytrope in a magnetic poloidal field was studied by Monaghan (1965). He found the differential equation that provides the magnetic field, solved it numerically for different polytropic indices and discussed the influence of the polytropic index n on the distribution of the magnetic field. Further, we will show that Monaghan's differential equation for the poloidal magnetic field is equivalent to an Euler equation and write its general solution. The Lane–Emden function present in this general solution will be substituted by the approximate analytic form proposed by Liu (1995), a fact that permits us a comparison between this solution and a numerical one.

2. BASIC EQUATION

To describe the equilibrium in a poloidal magnetic field we will use the equation of the hydromagnetic equilibrium, the Poisson equation, the Ampere law, the magnetic monopole equation and the polytropic relation, respectively:

$$\frac{\nabla P}{\rho} = -\nabla\phi + \frac{\vec{j}\times\vec{H}}{c\rho}, \qquad (1)$$

$$\nabla^2\phi = 4\pi G\rho, \qquad (2)$$

$$curl\vec{H} = \frac{4\pi}{c}\vec{j}, \qquad (3)$$

$$div\vec{H} = 0, \qquad (4)$$

$$P = K\rho^{1+\frac{1}{n}}, \qquad (5)$$

in which the notations are usual. Supposing that the star has axial symmetry and using the spherical coordinates (r,θ,ϕ) with $r = 0$ in the center of the star and $\theta = 0$ the symmetry axis, we obtain that $\frac{\partial}{\partial\phi} = 0$. Having in mind the representation of the general solution of (4) (see Chandrasekhar, 1961) and the simplifications due to the hypotheses of axial symmetry and poloidal magnetic field we will obtain that $\vec{H} = (H_r, H_\theta, 0)$, with :

$$H_r = \frac{1}{r^2\sin\theta}\frac{\partial S}{\partial\theta}, \quad H_\theta = -\frac{1}{r\sin\theta}\frac{\partial S}{\partial r} \qquad (6)$$

Taking curl of equation (1) and using (3) we will obtain :

$$curl\left(\frac{\vec{H}\times curl\vec{H}}{\rho}\right) = 0, \qquad (7)$$

Written in spherical coordinates it provides:

$$\frac{\partial}{\partial r}\left\{\frac{1}{\rho r\sin\theta}\left[\frac{1}{r\sin\theta}\frac{\partial^2 S}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial S}{\partial\theta}\right)\right]\frac{\partial S}{\partial\theta}\right\} - \frac{\partial}{\partial\theta}\left\{\frac{1}{\rho r\sin\theta}\frac{\partial S}{\partial r}\left[\frac{1}{r\sin\theta}\frac{\partial^2 S}{\partial r^2} + \frac{1}{r^3}\frac{\partial S}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial S}{\partial\theta}\right)\right]\right\} = 0 \qquad (8)$$

relation that will be used further to express the dependence of the magnetic field on radius. We shall assume, in the first approximation, that the magnetic field is weak, so that its presence will not modify the mass density distribution in the star. So, $\rho = \rho_0(r)$, where ρ_0 is known from the equilibrium of an unperturbed polytrope.

We will consider that the exterior magnetic field is dipolar and in the interior it is expressed by:

$$S(r,\theta) = A(r)\sin^2(\theta). \qquad (9)$$

In this situation the equation (8) becomes:

$$A\frac{d}{dr}\left[\frac{1}{\rho r^2}\left(\frac{2A}{r^2} - A''\right)\right] = 0 \qquad (10)$$

which can be integrated and gives:

$$\frac{2A}{r^2} - A'' = D\rho r^2 \qquad (11)$$

where D is an arbitrary constant of integration.

To facilitate the evaluation of the magnetic field we introduce the following transformations (Roxburgh,1966):

$$r = a\xi, \quad \rho_0 = \rho_c\theta_n^n, \quad A = D\rho_c a^4\gamma_n, \qquad (12)$$

$$a = \frac{K(n+1)\rho_c^{\frac{1}{n}-1}}{4\pi G}$$

in which we recognize the Emden variables (ξ,θ_n) and introduce a new dimensionless function $\gamma_n(\xi)$ proportional to the magnetic field. This substitution allows us to build the dimensionless form of (10). Using (6), (9) and (12) in (11) we get the following nonhomogeneous second order differential equation:

$$\frac{\partial^2\gamma_n}{\partial\xi^2} - \frac{2\gamma_n}{\xi^2} = -{\theta_n}^n\xi^2, \qquad (13)$$

where θ_n is the Lane–Emden function of index n, i.e. the solution of the Lane–Emden equation of order n:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta_n}{d\xi}\right) = -{\theta_n}^n, \qquad (14)$$

The boundary conditions for the equation (13) are :

$$\gamma_n(0) = \gamma_n'(0) = 0, \quad \left.\left(\xi\frac{d\gamma_n}{d\xi} + \gamma_n\right)\right|_{\xi=\xi_1} = 0 \qquad (15)$$

The first condition says that the magnetic field must be finite in the center of the star and the second in $\xi = \xi_1$ (with ξ_1 the polytropic radius, i.e. the first zero of the Lane–Emden function) reflects that at the surface of the star there is no discontinuity in the magnetic field.

3. THE GENERAL SOLUTION OF THE NONHOMOGENEOUS EQUATION

The equation (13) is a nonhomogeneous Euler equation. The unknown is function $\gamma_n(\xi) = -\frac{B_0^*(\xi)}{2}$ where $B_0^*(\xi)$ is the unknown function from the equation (27) in the Monaghan's (1965) paper. With the transformation $\xi = e^t$ it becomes an equation with constant coefficients:

$$\frac{d^2\tilde{\gamma_n}}{dt^2} - \frac{d\tilde{\gamma_n}}{dt} - 2\tilde{\gamma_n} = -\tilde{\theta_n}^n e^{4t}, \qquad (16)$$

where $\tilde{\gamma_n}(t) = \gamma_n(e^t)$, $\tilde{\theta_n}(t) = \theta_n(e^t)$. The fundamental system of solutions for (16) is:

$$\left\{e^{-t}, \quad e^{2t}\right\}, \qquad (17)$$

so the general solution is:

$$\tilde{\gamma_n}(t) = c_1(t)e^{-t} + c_2(t)e^{2t}, \tag{18}$$

where the functions $c_1(t)$ and $c_2(t)$ are determined from the following conditions:

$$c_1'(t)e^{-t} + c_2'(t)e^{2t} = 0, \tag{19}$$
$$-c_1'(t)e^{-t} + 2c_2'(t)e^{2t} = -\tilde{\theta_n}^n(t)e^{4t}$$

Solving the system (19) and substituting in (18) we find that the general solution is:

$$\tilde{\gamma_n}(t) = -\frac{1}{3}e^{2t}\int e^{2t}\tilde{\theta}_n^n(t)dt + \frac{1}{3}e^{-t}\int e^{5t}\tilde{\theta}_n^n(t)dt + K_1e^{2t} + K_2e^{-t} \tag{20}$$

where the constants K_1 and K_2, are determined using the boundary conditions (15).

Going back from the variable t to ξ we find the following expression for $\gamma_n(\xi)$:

$$\gamma_n(\xi) = -\frac{1}{3}\xi^2\int \xi\theta_n^n(\xi)d\xi + \frac{1}{3}\frac{1}{\xi}\int \xi^4\theta_n^n(\xi)d\xi + K_1\xi^2 + \frac{K_2}{\xi}. \tag{21}$$

Replacing in the integrands from (21) $\theta_n^n(\xi)$ with the left–hand member of the Lane–Emden equation and using the formula for integration by parts we found the following form for (21):

$$\gamma_n(\xi) = \xi^2\theta_n(\xi) - \frac{2}{\xi}\int \xi^2\theta_n(\xi)d\xi + K_1\xi^2 + \frac{K_2}{\xi} \tag{22}$$

where the constants K_1 and K_2 are determined using (25):

$$K_1 = -\frac{\xi_1}{3}\frac{d\theta_n(\xi_1)}{d\xi}, \quad K_2 = \left[12\int \xi^2\theta_n(\xi)d\xi\right]\Big|_{\xi=0} \tag{23}$$

We note that (21) and (23) involve the Lane-Emden function of index n. But as we know there are only three cases ($n \in \{0, 1, 5\}$) in which its exact form could be written (see Chandrasekhar, 1939). Further we will use an approximate form of it to be able to compare our results with the numerical results obtained using a routine of Runge–Kutta type.

4. CERTAIN SOLUTIONS FOR THE NON-HOMOGENEOUS EQUATION

If we substitute in (21) the exact solution of the Lane–Emden equation for $n = 0, 1$ we get two exact solutions for (13). For $n = 0$, i.e. incompressible medium: $\theta(\xi) = 1 - \frac{1}{6}\xi^2$ and we obtain the solution found by Ferraro (1954):

$$\gamma_0(\xi) = \xi^2 - \frac{\xi^4}{10} \tag{24}$$

For $n = 1$: $\theta(\xi) = \frac{\sin(\xi)}{\xi}$ we get the solution found by Monaghan (1965) multiplied by (-2) as we showed before:

$$\gamma_1(\xi) = \xi\sin\xi + 2\cos\xi - \frac{2\sin\xi}{\xi} + \frac{1}{3}\xi^2 \tag{25}$$

Table 1. The expressions of the functions f_i and F_i where $i \in \{1, 2\}$ and $n \in \{\frac{3}{2}, 2, 3\}$

n	$f_1(\xi, c, n)$	$f_2(\xi, c, n)$	$F_1(\xi, c, n)$	$F_2(\xi, c, n)$
$\frac{3}{2}$	$\frac{\xi^2}{(1+c\xi^2)^2}$	$\frac{\xi^4}{(1+c\xi^2)^3}$	$-\frac{1}{2}\frac{\xi}{c(1+c\xi^2)} + \frac{1}{2c\sqrt{c}}\arctan\sqrt{c}\xi$	$\frac{\xi}{4c^2(1+c\xi^2)^2} - \frac{5\xi}{8c^2(1+c\xi^2)}\frac{3\arctan\sqrt{c}\xi}{8c^2\sqrt{c}}$
2	$\frac{\xi^2}{1+c\xi^2}$	$\frac{\xi^4}{(1+c\xi^2)^2}$	$\frac{\xi}{c} - \frac{\arctan\sqrt{c}\xi}{c\sqrt{c}}$	$\frac{\xi}{c^2} + \frac{\xi}{2c^2(1+c\xi^2)} - \frac{3}{2}$
3	$\frac{\xi^2}{\sqrt{1+c\xi^2}}$	$\frac{\xi^4}{\sqrt{(1+c\xi^2)^3}}$	$\frac{1}{2}\frac{\xi\sqrt{1+c\xi^2}}{c} - \frac{1}{2}\frac{1}{c\sqrt{c}}\operatorname{arg\,sinh}\sqrt{c}\xi$	$\frac{\xi}{2c^2}\frac{3+c\xi^2}{\sqrt{1+c\xi^2}} - \frac{3}{2}\frac{\operatorname{arg\,sinh}\sqrt{c}\xi}{c^2\sqrt{c}}$

Table 2. The values of the parameters that appear in γ_n for certain polytropic indices

n	α	A_n	B_n	β	C_n	D_n	ξ_1	K_1	K_2
$\frac{3}{2}$	0.481	$\frac{1}{12}$	$1.1725\ 10^{-3}$	1.9821	$4.6184\ 10^{-4}$	$1.1\ 10^{-3}$	3.3538	0.2471	0
2	0.512	$\frac{1}{6}$	$2.5332\ 10^{-3}$	2.0043	$8.3218\ 10^{-4}$	$2.56\ 10^{-2}$	4.3529	0.1421	0
3	0.53	$\frac{1}{3}$	$4.648\ 10^{-3}$	2.1992	$5.5622\ 10^{-4}$	$2.745\ 10^{-2}$	6.8969	0.097	0

For other values of n we will use the approximate solution for (14) proposed by Liu (1995), i.e. for $0 \leq n \leq 5$, $n \neq 1$:

$$\theta_n(\xi) = -\alpha(1+B_n\xi^2)^{\frac{1}{1-n}} + (1+\alpha)(1+A_n\xi^2)^{\frac{1}{1-n}} + \frac{\alpha}{6}\xi^2(1+A_n\xi^2)^{\frac{n}{1-n}} + \frac{C_n\xi^{2\beta-1}}{(1+D_n\xi^\beta)^2} \quad (26)$$

where the constants A_n, B_n, β are defined by the following relations:

$$A_n = \frac{n-1}{6}, \quad B_n = \frac{n(n-1)}{(n-1)^2}\frac{6}{5}\left(\frac{4\alpha}{4+5\alpha}\right)^4 \quad (27)$$

$$\beta = 6.47 - 7.01\beta_1 + 5.53\beta_1^2 - 25.63\beta_2 + 49.42\beta_2^2 - 26.88\beta_2^3,$$

$$\beta_1 = \frac{1}{1+(n-5)^2}, \quad \beta_2 = \frac{1}{1+(n-3)^2} \quad (28)$$

and α, C_n, D_n are determined in each case such as the relation (26) approximates the Lane–Emden function with a maximum error $< 1\%$. We will specify their values when we use them. Let us substitute (26) in the general form of the solution (21) and use the formula for integration by parts. So we find:

$$\gamma_n(\xi) = -\alpha_n f_1(\xi, B_n, n) + (1+\alpha_n) f_1(\xi, A_n, n) + \frac{\alpha}{6} f_2(\xi, A_n, n) + \frac{C_n\xi^{2\beta+1}}{(1+D_n\xi^\beta)^2} + \frac{2\alpha_n}{\xi}F_1(\xi, B_n, n) - \frac{2(\alpha_n+1)}{\xi}F_1(\xi, A_n, n) - \frac{\alpha_n}{3\xi}F_2(\xi, A_n, n) + \frac{2C_n}{D_n\beta}\frac{\xi^{\beta+1}}{1+D_n\xi^\beta} - \frac{(\beta+2)C_n}{\beta D_n^2}\xi + \frac{2C_n(\beta+2)}{\xi\beta D_n^2}F_3(\xi, D_n, \beta) + K_1\xi^2 + \frac{K_2}{\xi} \quad (29)$$

in which we used the following notations:

$$f_1(\xi, c, n) = \xi^2(1+c\xi^2)^{\frac{1}{1-n}}, f_2(\xi, c, n) = \xi^4(1+c\xi^2)^{\frac{n}{1-n}},$$
$$f_3(\xi, c, \beta) = \frac{\xi}{1+c\xi^\beta}, \quad (30)$$

$$F_1(\xi, c, n) = \int f_1(\xi, c, n)d\xi, F_2(\xi, c, n) = \int f_2(\xi, c, n)d\xi, \quad (31)$$

$$F_3(\xi, c, \beta) = \int \frac{\xi d\xi}{1+c\xi^\beta} \quad (32)$$

Therefore to write the function γ_n we have to perform two indefinite integrals of binomial differential and one rational integral. The result of the integration of a binomial differential can be expressed using elementary functions only if the conditions of the Chebyshev theorem are fulfilled. This means restrictions on n, i.e. the polytropic index, so

$$n \in \{0, 2, \frac{q-1}{q}, \frac{p-2}{p}, \ where\ q, p \in \mathbf{Z}\} \quad (33)$$

Further we will restrict our discussion to the following values of the polytropic indices which are in the former set and are also of astrophysical interest $n \in \{\frac{3}{2}, 2, 3\}$. Let us evaluate the functions $f_i(\xi, c, n)$ and $F_i(\xi, c, n)$ for $i \in \{1, 2\}$. The results are listed in table 1.

For $F_3(\xi, c, \beta)$ we will use the following formulae (see Gradshteyn and Ryzhik,1970), for the values of the parameter β determined using (28) written as a quotient of two prime integer numbers $\frac{p}{q}$ and where we use the following substitution $x = c^{\frac{1}{p}}\xi^{\frac{1}{q}}$:

$$\int \frac{x^{n-1}}{1+x^{2k}}dx = \frac{1}{k}\sum_{\nu=1}^{k}\arctan\frac{x-\cos\frac{(2\nu-1)\pi}{2k}}{\sin\frac{(2\nu-1)\pi}{2k}} \sin\frac{n\pi(2\nu-1)}{2k} - \frac{1}{2k}\sum_{\nu=1}^{k}\ln\left(x^2 - 2x\cos\frac{(2\nu-1)\pi}{2k} + 1\right) \cos\frac{n\pi(2\nu-1)}{2k} \quad (34)$$

where $n < 2k$, i.e. $p+1 < q$ and q is an odd number, and:

$$\int \frac{x^{n-1}}{1+x^{2k+1}}dx = \frac{-1}{2k+1}\sum_{\nu=1}^{k}\ln(x^2 - 2x\cos\frac{(2\nu-1)\pi}{2k} + 1)\cos\frac{n\pi(2\nu-1)}{2k+1} + \frac{2}{2k+1}\sum_{\nu=1}^{k}\arctan\frac{x-\cos\frac{(2\nu-1)\pi}{2k+1}}{\sin\frac{(2\nu-1)\pi}{2k+1}} \sin\frac{n\pi(2\nu-1)}{2k+1} + (-1)^{n+1}\frac{\ln|1+x|}{2k+1} \quad (35)$$

when $n \leq 2k$, i.e. $p+1 \leq q-1$, q is an even number. Table 2. contains the values of the parameters present in the general solution (29) computed using (27), (28) or determined by Liu(1995).

So, all the entities from the relation (29) are now determined. Except the intricate formulae (34) and (35) all the other things could be easy computed. We have evaluated $\gamma_2(\xi)$ using (29) and compared the results with the numerical results. The maximum error was 0.00404638. We have represented these two solutions in Figure 1.

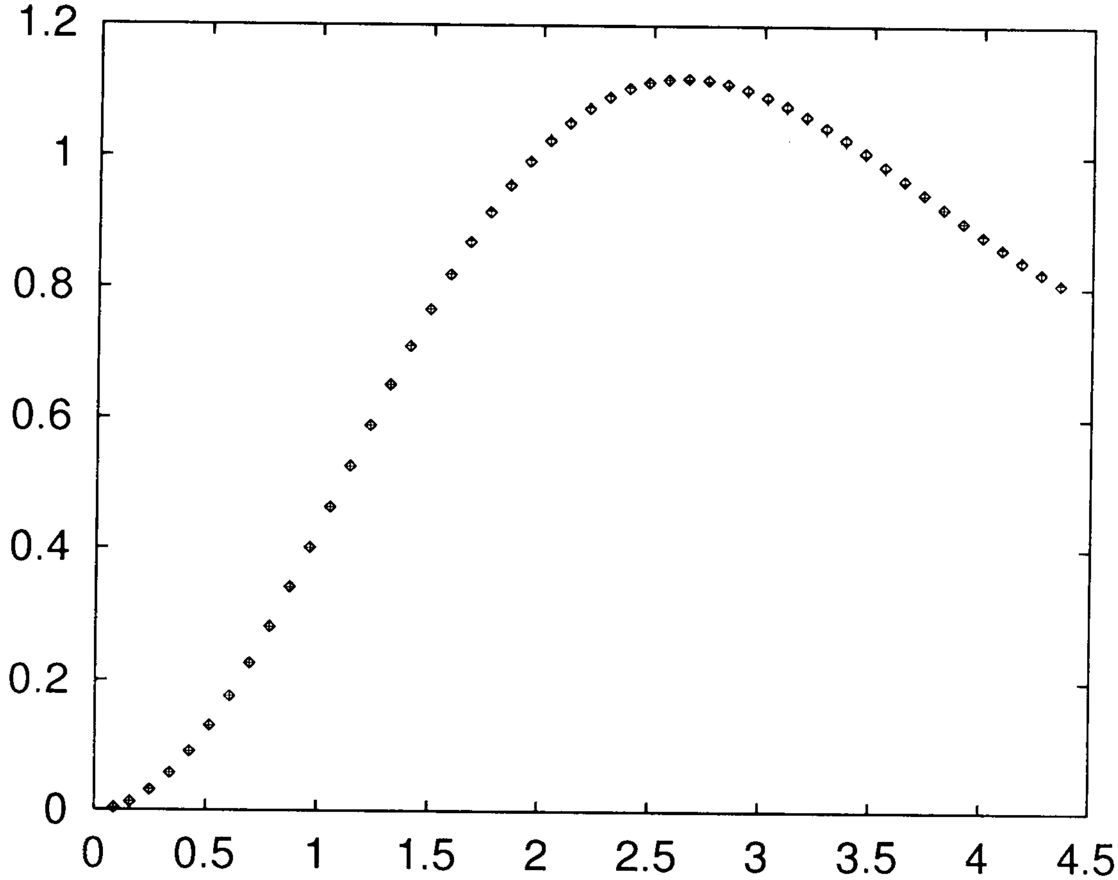

Fig. 1. *The poloidal magnetic field for the polytropic index n=2; the crosses are for numerical results and the dots for the analytical ones.*

A case treated separately by Liu (1995) was the polytrope $n = 1$. For this case the approximate solution for Lane–Emden equation is the following:

$$\theta_1(\xi) = -\alpha e^{-\frac{3}{10}\left(\frac{4\alpha}{4+5\alpha}\right)^4 \xi^2} + (1+\alpha)e^{-\frac{\xi^2}{6}} + \frac{\alpha}{6}\xi^2 e^{-\frac{\xi^2}{6}} + C_1 \xi^{2\beta_1 - 1} \quad (36)$$

where $C_1 = 1.27746\ 10^{-4}$, $\alpha = 0.455$ and $\beta_1 = 2.71254$ evaluated using (28). So, to find out the $\gamma_1(\xi)$ we have to substitute (36) in (29). We get:

$$\gamma_1(\xi) = \xi^2\theta_1(\xi) - \frac{2}{\xi}\left(-\alpha F_1(\alpha_1, \xi) + (1+\alpha)F_1(\alpha_2, \xi) + \frac{\alpha}{6}F_2(\alpha_2, \xi) + \frac{C_n \xi^{2\beta+2}}{2\beta+2}\right) + K_1\xi^2 + \frac{K_2}{\xi} \quad (37)$$

where we used the following notations:

$$F_1(c, \xi) = \int \xi^2 \mathrm{e}^{-c\xi^2} d\xi, \quad F_2(c, \xi) = \int \xi^4 \mathrm{e}^{-c\xi^2} d\xi,$$

$$\alpha_1 = \frac{3}{10}\left(\frac{4\alpha}{4+5\alpha}\right)^4, \qquad \alpha_2 = \frac{1}{6} \quad (38)$$

The integrals $F_i(c, \xi)$ with $i = 1, 2$ should be performed and yield:

$$F_1(c, \xi) = -\frac{1}{2}\frac{\xi \mathrm{e}^{-c\xi^2}}{c} + \frac{1}{4}\frac{\sqrt{\pi}\mathrm{erf}(\sqrt{c}\xi)}{c\sqrt{c}}$$

$$F_2(c, \xi) = -\frac{1}{2}\frac{x^3 \mathrm{e}^{-c\xi^2}}{c} - \frac{3}{4}\frac{\xi \mathrm{e}^{-c\xi^2}}{c^2} + \frac{3}{8}\frac{\sqrt{\pi}\mathrm{erf}(\sqrt{c}\xi)}{c^2\sqrt{c}} \quad (39)$$

The first zero of the Lane–Emden equation of 1^{st} order, i.e. ξ_1, is 3.141593. So, for the constants K_1 and K_2 we will use the relations (27) and obtain 0.334021 and 0 respectively. This approximate solution is represented in Figure 2., together with a numerical solution. In this case the maximum error is 0.0012780.

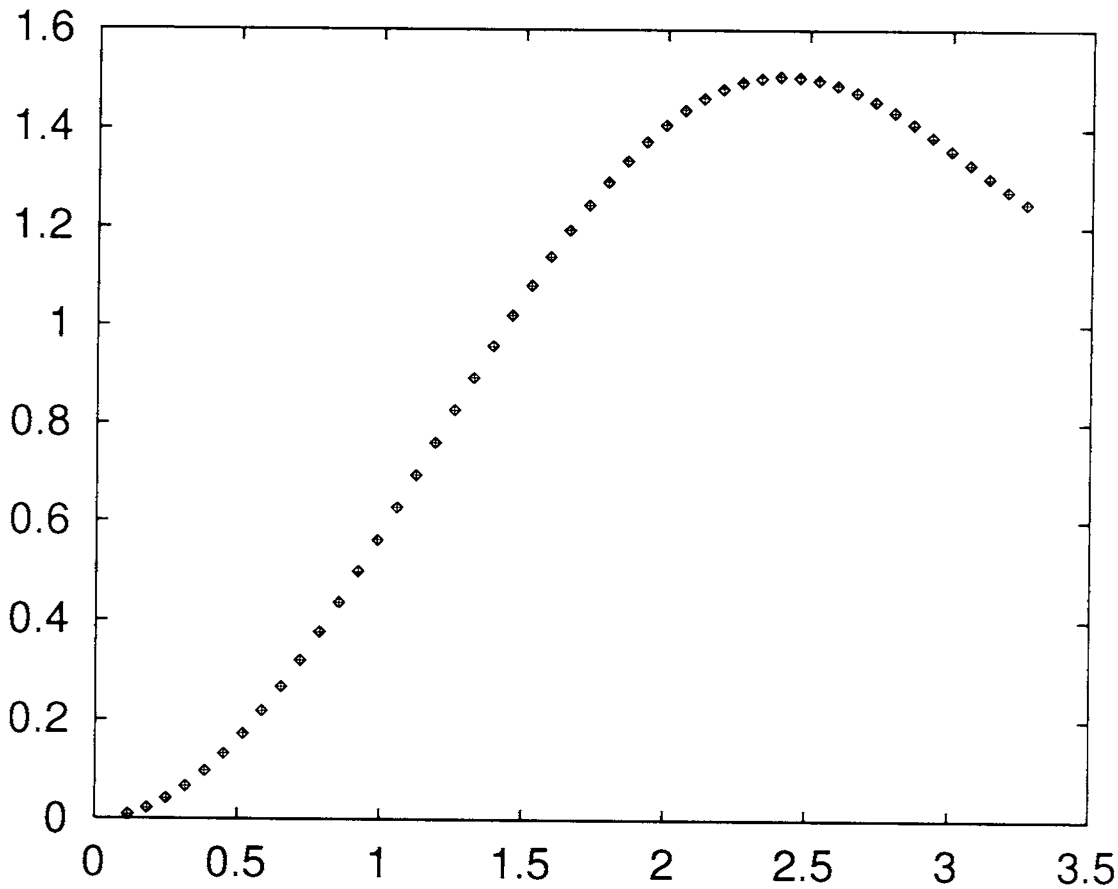

Fig. 2. *The poloidal magnetic field for the polytropic index n=1; the crosses are for numerical results and the dots for the analytical ones.*

5. CONCLUSION

The approximate solution for the Lane–Emden equation enables us to write the general solution of (26) for certain polytropic indices. This is a very easy way to obtain a good approximate distribution of the poloidal magnetic field with the depth inside the star. We focus our discussion on complete polytropes, i.e. in whole star the polytropic index is constant. This analytic solution could be used to consider composite polytropes. So far, we referred only to the magnetostatics. But, this result could also be useful if we are interested in small oscillations of the magnetic polytropes around the equilibrium.

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РАВНОТЕЖА ПОЛИТРОПА У ПОЛОИДНИМ МАГНЕТНИМ ПОЉИМА

К. Блага

Astronomical Institute of Romanian Academy
Astronomical Observatory Cluj
Cirećsilor Street 19, 3400 Cluj, Romania

УДК 524.3–1/–16
Оригинални научни рад

У овом чланку се проучава равнотежа политропа у полоидном магнетном пољу. Магнетно поље се налази као решење једне Ојлерове једначине за коју је могуће добити опште решење. Оно зависи од Лејн-Емденове функције. Овде се користи приближно аналитичко решење Лејн-Емденове једначине које је предложио Лиу да би се она написала експлицитно. За политропске индексе $n = 1, 2$ апроксимација је веома добра, максимална грешка је $< 0.5\%$.